THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: Shirley Elizabeth Mills

TITLE OF THESIS: Outliers in Life-Testing Distributions

DEGREE FOR WHICH THESIS WAS PRESENTED Doctor of Philosophy

YEAR THIS DEGREE GRANTED Fall, 1983

THE UNIVERSITY OF ALBERTA

Outliers in Life-Testing Distributions

by

Shirley Elizabeth Mills

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF STATISTICS AND APPLIED PROBABILITY

EDMONTON, ALBERTA

Fall, 1983

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled

Outliers in Life-Testing Distributions

submitted by Shirley Elizabeth Mills in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

DEDICATION

To Kathryn

ABSTRACT

Much work has been done on outliers in normal populations and recently in exponential populations. Here we extend the study to the examination of outliers in three competing life-testing distributions. Assuming the exchangeable model of random variables, we examine the concepts of outlier-prone and outlier-resistant families as they apply to the Gamma, Lognormal, and Weibull families of density functions. We examine also the detection of outliers for these distributions and the estimation of parameters in the presence of one or more spurious observations.

ACKNOWLEDGEMENTS

I would like to thank Dr. J.R. McGregor, whose patience, guidance, encouragement and advice have brought this work to fruition. I would also like to thank Dr. B.K. Kale for suggesting research on the outlier problem, Mr. Walter Aiello for programming assistance and Mrs. Christine Fischer for typing this manuscript.

To my husband Clayton, who has been a source of encouragement and advice during preparation of this thesis and who has dealt with the "outlier problem" firsthand, I offer special thanks.

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LIST OF SYMBOLS

NOTATION	DESCRIPTION	PAGE
n	sample size	6
F	distribution function	6
G	distribution function	6
m	number of outliers	6
τ	location parameter	6
θ	scale parameter	6
S_n	sample of size n	6
$f(x;\underline{\theta})$	probability density function of target population P	6
$f(x;\underline{\xi})$	probability density function of spurious population Q	7
$L(\underline{x};\underline{\theta},\underline{\xi})$	likelihood function for the exchangeable model	7
$N(\tau,\theta^2)$	normal distribution with mean τ and variance θ^2	7
$EXP(\theta)$	exponential distribution with mean θ	7
k^*	coefficient of spuriosity	
I	set of spurious observations	8
$\mathcal{I}$	set of all $\binom{n}{m}$ possible sets of spurious observations	8
$P(k,n\|F)$	$P\{x_{(n)}-x_{(n-1)}>k(x_{(n-1)}-x_{(1)})\|x_i \sim F\}$	9
$\Pi_1(k,n\|\mathcal{F})$	$\sup_{F\in\mathcal{F}} P(k,n\|F)$	10
$C(\xi,\theta)$	Cauchy distribution centered at ξ, scale θ	12

$u(r;n,k^*)$	$P\{X_{(r)}$ is spurious in a sample of size $n\}$	14
$\Psi(x)$	$dG(x)/dF(x)$	14
$T_{m,n}$	$\dfrac{\sum_{i=1}^{n-m} x_{(i)} + mx_{(n-m)}}{n-m+1}$	16
$\hat{\theta}_{m,n}$	$\dfrac{\sum_{i=1}^{n-m} x_{(i)}}{n-m}$	17
T_t	$\dfrac{\sum_{i=1}^{n-1} x_{(i)}}{n}$	17
$D_{k^*}(\underline{x})$	$\begin{cases} T_{0,n} & \text{if } x_{(n)} - x_{(n-1)} < k(x_{(n-1)}-x_{(1)}) \\ T_t & \text{otherwise} \end{cases}$	17
$T_{k^*}(\underline{x})$	$\begin{cases} T_{0,n} & \text{if } x_{(n)} < C\bar{x} \text{ , } C > 0 \\ T_t & \text{otherwise} \end{cases}$	17
$\hat{\theta}_i$	$\sum_{\substack{j=1 \\ j\neq i}}^{n} x_j/n$	18
$\hat{\theta}_{CK}$	$\sum_{i=1}^{n} \omega_i\hat{\theta}_i,\ \omega_i = \dfrac{2r_i}{n(n+1)}$, r_i = rank of x_i	18
$GAM(\theta,\eta,\tau)$	$f(x,\theta,\eta,\tau) = \dfrac{(x-\tau)^{\eta-1}e^{-(x-\tau)/\theta}}{\theta^{\eta}\Gamma(\eta)}$, $x > \tau$, $\theta,\ \eta > 0,\ -\infty < \tau < \infty$	22
η	shape parameter	22
χ^2_ν df	Chi-square density with ν degrees of freedom	24
$h(x)$	HF; hazard function = $\dfrac{f(x)}{1-F(x)}$	24
E	event that $x_{(n)}$ is a (k,n)-outlier	26
$P(E\|\eta,k^*)$	$P(k,n\|L)$	26
$I_r(\eta,k^*)$	$\int_E (n-1)! \prod_{i\neq r} f(x_{(i)};\eta)(f(x_{(r)};k^*\eta)dx_{(1)}\cdots dx_{(r)}$	27
$u(r;n,\eta,k^*)$	$P(X_{(r)}$ is the spurious observation)	27

$I_n'(\eta,k^*)$		28
$\vartheta(\nu,t)$		28
$\hat{\theta}_{MLE}$	$\bar{x}/\eta$ maximum likelihood estimator of scale	38
$\tilde{x}$	$(\prod_{i=1}^{n} x_i)^{1/n}$	38
$\psi(z)$	Euler's psi function $\frac{d}{dz} \ln \Gamma(z)$	38
γ	Euler's constant .5772157	38
$\hat{\eta}_{MLE}$	Maximum likelihood estimator of shape	38
$\hat{\eta}_M$	Moment estimator of shape	39
$\hat{\theta}_M$	Moment estimator of scale	39
$\hat{\hat{\eta}}$	Lilliefors' estimator of shape η	39
$\hat{\hat{\theta}}$	Lilliefors' estimator of scale θ	39
η^*	Thom's estimator of shape η	40
θ^*	Thom's estimator of scale θ	40
M	$\ln(\bar{x}/\tilde{x}) = -\ln S_1$	40
$L(\underline{x};\theta,\eta,k^*,I)$		42
$\Lambda(\mu,\sigma)$	lognormal density $f(x;\mu,\sigma) = \frac{1}{\sigma x\sqrt{2x}} \exp\{-\frac{1}{2}\frac{(\ln x-\mu)^2}{\sigma^2}\}$	53
$L(\underline{x};\sigma,k^*)$		59
$\phi(z)$	standard normal p.d.f.	71
$\Phi(z)$	standard normal distribution function at z	71
$u(r;n,\sigma,k^*)$	$P\{X_{(r)}$ is the spurious observation$\}$	75
$\hat{\mu}$	M.L.E., B.L.I.E. of μ, ; $\hat{\mu} = \bar{w} = \frac{\sum_{i=1}^{n} \ln x_i}{n}$	78
$\hat{\sigma}^2$	M.L.E. of σ^2	78
$\tilde{\sigma}^2$	B.L.I.E. of σ^2	78

$L(\underline{x};\mu,\mu_1,\sigma,I)$		81
$\hat{\mu}_{het}$	M.L.E. of μ under exchangeable model	82
$\hat{\mu}_{1_{het}}$	M.L.E. of μ_1 under exchangeable model	82
$\hat{\sigma}^2_{het}$	M.L.E. of σ^2 under exchangeable model	82
$WEI(\theta,\eta,\tau)$	Weibull distribution $f(x;\theta,\eta,\tau) = \frac{\eta(x-\tau)^{\eta-1}}{\theta^\eta} \exp\{-(\frac{x-\tau}{\theta})^\eta\}$	86
$EV_I(\xi,b)$	Extreme-Value distribution $F(y) = 1 - \exp\{-\frac{\exp(y-\xi)}{b}\}$	88
$Q(k,n\mid\eta)$		91
$L(\underline{x};\eta,k^*)$		92
$u(r;n,k^*)$		99
$L(u,p)$	Laplace transform of $e^{-\nu^u}$, Re $u > 0$	100
$\hat{\eta}$	M.L.E. of η	109
$\hat{\theta}$	M.L.E. of θ	109
$\tilde{b}_{BLIE}$	Mann and Fertig's (1973) BLIE of b	109
$\hat{b}_k$	Mann and Fertig's adaptation of Hassanein's estimator of b	110
b^*	Unbiased form of $\hat{b}_k$	110
b^*_{BLIE}	BLIE based on b^*	110
r	size of censored sample	110
$\hat{b}_B$	Bain's estimator of b	110
$k_{r,n}$		110
$\hat{\eta}_B$	Bain's estimator of η	111
$\hat{b}_s$	Englehardt and Bain's (1973) estimator of b	111
$\tilde{b}_{MF}$	Mann and Fertig's (1975) BLIE adaptation of $\hat{b}_s$	111
$\ell_{k,n}$		112

$\hat{b}_{EB}$	Englehardt and Bain's (1977) estimator of b	112
k_n		112
b^*_{EB}	BLIE based on $\hat{b}_{EB}$	113
$\hat{b}_{MN}$	Menon's estimator of b	113
$\hat{\xi}_{MN}$	Menon's estimator of ξ	113
$\hat{\eta}_{MN}$	Menon's estimator of η	113
$\hat{\theta}_{MN}$	Menon's estimator of θ	113
$\hat{\eta}_D$	Dubey's estimator of η	113
$\hat{b}_{MS}$	Murthy-Swartz estimator of b	114
$\hat{b}_{RA}$		117
$\hat{b}_{RW}$		117
$\hat{b}_{RS}$		117
$\hat{\xi}_{RA}$		117
$\hat{\xi}_{RW}$		117
$\hat{\xi}_{RS}$		117
$\hat{\mu}_A$	A-rule estimator for mean of normal distribution	129
$\hat{\sigma}^2_A$	A-rule estimator for variance of normal distribution	129
$\hat{\mu}_W$	W-rule estimator for mean of normal distribution	130
$\hat{\mu}_S$	S-rule estimator for mean of normal distribution	130
$\hat{\sigma}^2_W$	W-rule estimator for variance of normal distribution	130
$\hat{\sigma}^2_S$	S-rule estimator for variance of normal distribution	130
$\zeta(x)$	Reimann's zeta function	185

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CHAPTER I

OUTLIERS AND OUTLIER-PRONENESS

What is an outlier and what is the outlier problem? To quote Ferguson (1961):

> In a sample of moderate size taken from a certain population it appears that one or two values are surprisingly far way from the main group. The experimenter is tempted to throw away the apparently erroneous values, and not because he is certain that the values are spurious. ... It is rather because he feels that other explanations are more plausible, and that the loss in the accuracy of the experiment caused by throwing away a couple of good values is small compared to the loss caused by keeping even one bad value.

Barnett and Lewis (1978) indicate that in the light of developments in outlier methodology over the last 15 years, Ferguson's formulation may be unduly restrictive. Outlying values need not be "bad" or "erroneous"; in fact they may be welcomed as indicating a useful treatment or a strain that was unexpectedly good.

1.1 What is an outlier?

What do we mean by an "outlier"? We shall use Barnett and Lewis' definition (1978): "an outlier in a set of data (is) ... an observation (or subset of observations) which appears to be inconsistent with the remainder of that set of data." We use the term "outlier" to characterize an observation that stands out in contrast to the rest of the data and is not consistent with what our mind feels constitutes a reasonable data set nor with our initial view of an appropriate probability model to describe the generation of our data. An observation is "spurious" if it is statistically unreasonable on the basis of some prescribed probability model. This would include an observation known to be generated by a different probability model; however a spurious observation will not necessarily show up as an outlier.

1.2 The outlier problem

Experimental scientists and others who deal with data are forced to make decisions about outliers - whether or not to include them in analysis, whether to make allowances for them on some compromise basis, etc. What worries an experimenter is whether or not some observations are genuine members of the main population. If these observations appear in the midst of the data they may not be conspicuous and are unlikely to distort inferences seriously. However, should the outlier appear extreme, it could create problems in attempts to represent the population and it could contaminate estimates and tests of population parameters.

Once we decide that outliers exist in a data set, we must decide on how to react to them. Methods to support outright rejection or to adjust values, prior to processing the principal mass of data will be related to any postulated model for that population. We are concerned with whether the extreme values are so extreme as to be unreasonable under our original model. This may indicate the data contains a "mistake" (that should be rejected or corrected) or it may indicate an alternative model for which the complete data set appears as a homogeneous sample. The outlier may be a 'foreign' random influence in an otherwise homogeneous data set - interesting to study in itself or only serving to obstruct analysis of the main data mass.

An assessment of the discordancy of some outliers is just the first stage of the study of outliers. Once an outlier is judged discordant we may:

1) decide to reject (or correct) it and analyse the remaining (modified) data on the original model,

2) decide to modify the model to incorporate the outliers in a non-discordant manner,

3) refine the analysis of the entire data set to accommodate outliers (i.e. render the analysis relatively impervious to their presence),

4) focus attention on the discordant outliers because they identify factors of unexpected value.

As examples of each of these situations, consider the following:

Fifteen residuals (about a simple model) of observations of the vertical semi-diameter of Venus, in seconds, made by Lt. Herndon in 1846 were

-0.30	+0.06	-0.05
-0.44	+0.63	+0.20
+1.01	-0.13	+0.18
+0.48	-1.40	+0.39
-0.24	-0.22	+0.10

Chauvenet (1863) declared -1.40 and +1.01 outliers. We may choose to <u>reject</u> them as "gross errors"; <u>incorporate</u> them by changing our model to a non-normal one or; <u>accommodate</u> them by using a robust estimation technique that employs Winsorization or an α-trimmed sample.

Consider also the data described by Pearson and Pearson (1931) on the capacity (in c.c.) of a sample of 17 male Moriori skulls.

1230	1318	1380	1420	1630	1378
1348	1380	1470	1445	1360	1410
1540	1260	1364	1410	1545	

The observation 1630 is suspicious and as such may be rejected as a measurement/recording error; or we may incorporate it in a non-discordant way by assuming a non-normal model (since biological data often require skew distributions); or identification of the outlier may reflect the presence of a small number of another species in the population being studied.

1.3 Models for discordancy

There are several possible models for discordancy.

i) Deterministic: this covers the cases of outliers caused by obvious identifiable gross measurement errors, etc. If the data set $\{x_i\}_{i=1}^{n}$ contains one observation, x_i, clearly resulting from measurement/recording error, we reject H: all $x_i \in F$ in favor of H_1: all $x_j \in F$ $(j \neq i)$ and x_i is different (requiring rejection or correction).

ii) Inherent: here we reject H: all observations are from F in favor of H_1: all observations are from G, where G has more or different inherent variability than F.

iii) Mixture: this model allows contamination of the sample by a few members of a population other than F. We reject H: all $x_i \in F$ in favor of H_1: all $x_i \in pF+(1-p)G$. Thus outliers reflect probability 1-p that observations arise from G.

iv) Slippage: here H_1 states all observations apart from some small number m arise independently from the initial model F indexed by location and scale parameters τ and θ, while the remaining m are independent observations from a modified version of F in which τ or θ is shifted. Which observation(s) come(s) from the shifted distribution is not specified. In most published work, F represents the normal distribution.

v) Exchangeable: this is an extension of the slippage alternative. A set of n observations $S_n = \{x_1,...,x_n\}$ ideally comes from a target population P with probability density function (p.d.f.) $f(x;\underline{\theta})$. However the suspicion is that one of the observations, say x_i, is not

from the target population P but from a different population Q with p.d.f. $f(x;\underline{\xi})$. Prior to the experiment, there is no way to identify the possible (at most one spurious observation). Hence, a priori, each observation is equally likely to be the discordant one. Thus the random variables $X_1,\ldots,X_n$ are not independent but are exchangeable. For one outlier, the likelihood of the sample is given by

$$L(\underline{x}|\underline{\theta},\underline{\xi}) = \frac{1}{n}\sum_{i=1}^{n} f(x_i;\underline{\xi}) \prod_{j\neq i} f(x_j;\underline{\theta}) \ .$$

Definition 1.3.1: Let $\mathcal{F}$ be a family of absolutely continuous univariate distribution functions $\{F(x;\underline{\theta})\}$ indexed by a parameter $\underline{\theta} \in \Theta$. Then the random variables $X_1,\ldots,X_n$ are said to be exchangeable random variables based on $\mathcal{F}$ if their joint p.d.f. is of the form

$$\frac{1}{n}\sum_{r=1}^{n} \prod_{i\neq r} F'(x_i;\underline{\theta}_1)F'(x_r;\underline{\theta}_2), \quad \underline{\theta}_1,\underline{\theta}_2 \in \Theta$$

where $F'(x;\underline{\theta}) = \frac{\partial}{\partial x} F(x;\underline{\theta})$.

Anscombe (1960) used this model for $f(x;\underline{\theta})$ as $N(\tau,\theta^2)$ and $f(x;\underline{\xi})$ as $N(\tau+a\theta,\theta^2)$. Guttman and Smith (1969,1971) and Guttman (1973a) used it for $f(x;\underline{\theta})$ as $N(\tau,\theta^2)$ and $f(x;\underline{\xi})$ as $N(\tau+a,\theta^2)$. Kale and Sinha (1971), Joshi (1972b), Kale (1974), Chikkagoudar and Kunchur (1980) and Rauhut (1982) have used it for $f(x;\underline{\theta})$ as $EXP(\theta)$ and $f(x;\underline{\xi})$ as $EXP(\theta/k^*)$, $0 < k^* \leq 1$.

For m outliers, we assume $x_{i_1},x_{i_2},\ldots,x_{i_{n-m}}$ come from a target

population P with p.d.f. $f(x;\underline{\theta})$, while $x_{i_{n-m+1}},\ldots,x_{i_n}$ come from populations $Q_1,Q_2,\ldots,Q_m$ with p.d.f. $f(x;\underline{\xi}_1),\ldots,f(x;\underline{\xi}_m)$. Some or all of the $\underline{\xi}_i$, $i = 1,\ldots,m$ may be identical (i.e. $\underline{\xi}_i = \underline{\xi}$). The association of the different observations with the different distributions is assumed to occur at random. For the case of all $\underline{\xi}_i = \underline{\xi}$, $i = 1,\ldots,m$, the likelihood of the sample is

$$L(\underline{x}|\underline{\theta},\underline{\xi}) = \frac{1}{\binom{n}{m}} \sum_{I\in\mathcal{J}} \prod_{i\in I} f(x_i;\underline{\xi}) \prod_{j\notin I} f(x_j;\underline{\theta})$$

where $I = (i_1,i_2,\ldots,i_n)$ is a selection of m integers from $\{1,2,\ldots,n\}$ and $\mathcal{J}$ is the set of all $\binom{n}{m}$ possible such choices. Certain relationships must exist between $f(x;\underline{\theta})$ and $f(x;\underline{\xi})$ for it to be reasonable that the suitability of the model will be reflected in outliers. For the case of one outlier, we need the discordant observation from Q to show up at one of the sample extremes. The exchangeable model also assumes that the maximum number of possible outliers is known.

1.4 The concept of outlier-proneness

Related to model development, though not actually a model to describe the occurrence of outliers, is a concept introduced by Neyman and Scott (1971) and furthered by Green (1974,1976) and Kale (1975b, 1975c, 1976). These papers considered a method of distinguishing between families of distributions by examining the extent to which they are liable to exhibit outliers.

Let S_n be a sample of $n \geq 3$ observations and let $x_{(1)},\ldots,x_{(n)}$ be the order statistics for this sample.

Definition 1.4.1: For a positive number k, $x_{(n)} \in S_n$ is a k-outlier on the right if $x_{(n)} - x_{(n-1)} > k\{x_{(n-1)}-x_{(1)}\}$. The definition of a k-outlier on the left is analogous (i.e $x_{(1)} \in S_n$ is a k-outlier on the left if $x_{(2)} - x_{(1)} > k\{x_{(n)}-x_{(2)}\}$.

Let $P(k,n|F)$ denote the probability that a sample of n observations from a distribution F will contain a k-outlier on the right.

Then, for jointly distributed random variables

$$P(k,n|F) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_{(n-1)}}\int_{(k+1)x_{(n-1)}-kx_{(1)}}^{\infty} g(x_{(1)},x_{(n-1)},x_{(n)})\, dx_{(n)}dx_{(1)}dx_{(n-1)}$$

where $g(x_{(1)},x_{(n-1)},x_{(n)})$ is the marginal joint p.d.f. of $X_{(1)}, X_{(n-1)}, X_{(n)}$ given by

$$g(x_{(1)},x_{(n-1)},x_{(n)}) = \int_{x_{(1)}}^{x_{(n-1)}} \cdots \int_{x_{(1)}}^{x_{(3)}} f(y_1,\ldots,y_n)dy_2\ldots dy_{n-2}$$

where f is the joint p.d.f. of $X_1,\ldots,X_n$.

For the special case of i.i.d. random variables we would have

$$P(k,n|F) = n(n-1)\int_{-\infty}^{\infty} dF(x)\int_{x}^{\infty} \left\{F\left(\frac{kx+y}{k+1}\right) - F(x)\right\}^{n-2} dF(y).$$

Let $\mathcal{F}$ be the family of distributions and let $\Pi_1(k,n|\mathcal{F})$ be the least upper bound of probabilities $P(k,n|F)$ for $F \in \mathcal{F}$.

Definition 1.4.2: A family $\mathcal{F}$ of distributions is (k,n)-outlier-prone on the right if $\Pi_1(k,n|\mathcal{F}) = 1$.

Definition 1.4.3: A family $\mathcal{F}$ of distributions is (k,n)-outlier-resistant on the right if $\Pi_1(k,n|\mathcal{F}) < 1$.

Definition 1.4.4: If a family $\mathcal{F}$ of distributions is (k,n)-outlier-prone on the right $\forall k > 0$, $\forall n > 2$ it is outlier-prone completely on the right (o.p.c.r.).

Definition 1.4.5: If a family $\mathcal{F}$ of distributions is (k,n)-outlier-resistant on the right $\forall k > 0$, $\forall n > 2$ it is outlier-resistant completely on the right. (o.r.c.r.).

Theorem 1.4.6: (Green (1974))

If $\mathcal{F}$ is a family of distributions and S_n is a random sample of n i.i.d. observations from $F \in \mathcal{F}$, then $\mathcal{F}$ is outlier-prone completely on the right (o.p.c.r.) iff $\mathcal{F}$ is (k,n)-outlier-prone on the right for some $k > 0$, $n > 2$.

For i.i.d. observations, Theorem 1.4.6 shows that it is not necessary to distinguish between the concepts of "(k,n)-outlier-prone on the right" and "outlier-prone completely on the right". This leads to the following definitions and theorem for cases in which the observations are i.i.d.

Definition 1.4.7: With respect to a random sample of n i.i.d. observations a family $\mathcal{F}$ is outlier-prone on the right (o.p.r.) iff it is (k,n)-outlier-prone on the right for some $k > 0$ and $n > 2$, or, equivalently, iff it is outlier-prone completely on the right.

Definition 1.4.8: With respect to a random sample of n i.i.d. observations a family $\mathcal{F}$ is outlier-resistant on the right (o.r.r.) iff it is not o.p.r.

Theorem 1.4.9: With respect to a random sample of n i.i.d. observations a family $\mathcal{F}$ is o.r.r. iff $\Pi_1(k,n|\mathcal{F}) < 1$ for some $k > 0$, $n > 2$, $F \in \mathcal{F}$.

Proof: i) Assume that family $\mathcal{F}$ is o.r.r. . Then $\mathcal{F}$ is not o.p.r. .

Therefore $\Pi_1(k,n|\mathcal{F}) \neq 1 \quad \forall k > 0, \forall n > 2.$

i.e. There exists at least one $k' > 0$, $n' > 2$, $F \in \mathcal{F} \ni$.

$$\sup_{F \in \mathcal{F}} P(k',n'|F) < 1 \ .$$

ii) Assume $\Pi_1(k,n|\mathcal{F}) < 1$ for some $k > 0$, $n > 2$, $F \in \mathcal{F}$. Then

$\Pi_1(k,n|\mathcal{F}) \neq 1 \quad \forall k > 0, \quad \forall n > 2$.

Therefore $\mathcal{F}$ is not o.p.r.

From Definition 1.4.8 it follows that $\mathcal{F}$ is o.r.r. .

We will use the same definition of a (k,n)-outlier on the right when S_n represents n observations from the exchangeable model.

The implication is that for an outlier resistant family we are justified in seeking out and eliminating outliers. On the other hand, outlier-prone families should be used to model cases where outliers are common and in these cases we should seek ways of accommodating outliers.

Neyman and Scott (1971) showed that in general families differing only in location or scale parameters are outlier-resistant. Thus we may limit studies to subfamilies with standard values for location and scale. Both the family $N(\tau,\theta^2)$ or normal distributions (with mean τ and variance θ^2) and the family $C(\xi,\theta)$ of Cauchy distributions (centered at ξ and having scale parameter θ) are outlier-resistant.

CHAPTER II

SURVEY OF THE LITERATURE

Most studies of the outlier-problem assume an initial distribution that is either exponential or normal.

2.1 Detection of outliers in exponential and normal families using the exchangeable model.

A semi-Bayesian approach was used by Kale (1969), Kale and Sinha (1971), Chikkagoudar and Kunchur (1980), and Rauhut (1982) with respect to the one-parameter exponential family. It was assumed that $X_1,\ldots,X_n$ were such that $n-1$ observations were from $f(x;\theta_1) = \frac{1}{\theta_1}\exp\{\frac{-x}{\theta_1}\}$, $x \geq 0$, $\theta_1 > 0$ while one of the X_i's was distributed as $f(x;\theta_2)$, $\theta_2 \geq \theta_1$ (i.e. $\theta_2 = \theta_1/k^*$, $0<k^*\leq 1$). *A priori*, each X_i was equally likely to be distributed as $f(x;\theta_2)$. Kale and Sinha (1971) calculated $u(r;n,k^*)$, the probability that $X_{(r)}$, the r^{th} order statistic, corresponds to the spurious observation distributed as $f(x;\theta_2)$. It was shown that

$$u(r;n,k^*) = \frac{k^*\Gamma(n)\Gamma(n-r+k^*)}{\Gamma(n+k^*)\Gamma(n-r+1)}$$

was monotone increasing in r and hence $X_{(n)}$ had maximum posterior probability of being an outlier. Mount and Kale (1973) generalized this result for the case of $n-1$ observations with distribution function F and one with distribution function G where F and G are stochastically ordered ($G<F$), where *a priori* each observation is equally likely to be the spurious one, and where $\Psi(x) = \frac{dG}{dF}$ is monotone increasing in r. Kale (1974a) generalized these results for the case of m (≥ 1) possible outliers, where *a priori* each group of m observations $(X_{i_1},X_{i_2},\ldots,X_{i_m})$ is equally likely to come from distribution function G. Then $(x_{(n-m+1)},\ldots,x_{(n)})$ has maximum

posterior probability of corresponding to the set of spurious observations, provided Ψ is monotone increasing. Kale restricted the distributions to the single-parameter exponential family. In another paper, Kale (1974b) gave a completely Bayesian approach where $n-m$ observations were distributed as $f(x;\theta)$, m_1 observations were from $f(x;\theta_j)$, $j = 1,2,\ldots,m_1$, $\theta_j \leq \theta$, and m_2 observations were from $f(x;\theta_\ell)$, $\ell = 1,2,\ldots,m_2$, $\theta_\ell \geq \theta$, $m = m_1+m_2$ and f belonged to the single-parameter exponential family. Under the exchangeable model, it was shown that $\{(x_{(1)},\ldots,x_{(m)}),(x_{(n-m+1)},\ldots,x_{(n)})\}$ had maximum posterior probability of being the set of spurious observations.

For the case of the normal family of distributions, a completely Bayesian approach has been used by Box and Tiao (1968) where $f(x;\underline{\theta})$ is $N(\tau,\theta^2)$ and $f(x;\underline{\theta}_j)$ is $N(\tau,k^*\theta^2)$, $k^* \geq 1$ (Model A) and by Dempster and Rosner (1971) where $f(x;\underline{\theta})$ is $N(\tau,\theta^2)$ and $f(x;\underline{\theta}_j)$ is $N(\tau_j,\theta^2)$, $j = 1,2,\ldots,m$, $\tau_j \geq \tau$ (Model B). For $m = 1$, Model B has been handled as a slippage test by Ferguson (1961), Kudo (1956) and Paulson (1952).

2.2 Estimation in the presence of outliers.

Anscombe (1960) suggested a premium-protection approach (see Appendix 1) to estimation that has subsequently been used by Kale and Sinha (1971), Joshi (1972b), Chikkagoudar and Kunchur (1980), and Rauhut (1982) for estimation of the mean in the single-parameter exponential distribution and by Guttman and Smith (1969, 1971, 1973a) and Veale and Huntsberger (1969) for estimation in the normal distribution.

2.2.1 Estimation for the single-parameter exponential distribution.

Under homogeneity, the optimal estimator of θ is $T_{o,n} = \dfrac{\sum_{i=1}^{n} x_i}{n+1}$. However, under the exchangeable model with one outlier,

$$\mathrm{MSE}(T_{o,n}|k^*) = \theta^2\left\{\frac{1}{n+1} + \frac{2}{n+1}\left(\frac{1-k^*}{k^*}\right)^2\right\} \to \infty \quad \text{as} \quad k^* \to 0.$$

Among restricted L-type estimators $T(\underline{\ell}) = \sum_{j=1}^{n-m} \ell_j x_{(j)}$ which ignore the largest m observations, Kale and Sinha (1971) and Veale and Kale (1972) advocated the one-sided Winsorized mean

$$T_{m,n} = \frac{\sum_{i=1}^{n-m} x_{(i)} + m x_{(n-m)}}{n-m+1}$$

for which $\mathrm{MSE}(T(\underline{\ell})|k^*=1)$ is minimum and $\mathrm{MSE}(T_{m,n}|k^*<1)$ shows a gain in efficiency relative to $T_{o,n}$ for k^* sufficiently small. Joshi (1972(b)) considered choice of m and showed for k^* small, substantial gains in relative efficiency are possible. Samples up to size $n = 20$ were considered.

Kale (1974a) considered maximum likelihood estimation (M.L.E.) for the exchangeable model with m possible upper outliers and obtained $\hat{\theta}_{m,n} = \frac{\sum_{i=1}^{n-m} x_{(i)}}{n-m}$, the trimmed mean, as MLE of θ. In comparing the trimmed mean $\hat{\theta}_{1,n} = \frac{\sum_{i=1}^{n-1} x_{(i)}}{n-1}$ with the Winsorized estimator $T_{1,n} = \frac{\sum_{i=1}^{n-1} x_{(i)} + x_{(n-1)}}{n}$ for the case of a single upper outlier, it was shown that

$$\mathrm{MSE}(\hat{\theta}_{1,n} | k^*=1) > \mathrm{MSE}(T_{1,n} | k^*=1)$$

but $\lim_{k^* \to 0} \mathrm{MSE}(\hat{\theta}_{1,n} | k^*) = \frac{1}{n-1} < \lim_{k^* \to 0} \mathrm{MSE}(T_{1,n} | k^*)$. Thus $\hat{\theta}_{1,n}$ provides more protection than $T_{1,n}$ but for a higher premium (See Appendix I).

Rauhut (1982) suggested two 'testimators":

$$D_{k^*}(\underline{x}) = \begin{cases} T_{o,n} & , \text{ if } x_{(n)} - x_{(n-1)} < k\{x_{(n-1)} - x_{(1)}\} \\ \frac{\sum_{i=1}^{n-1} x_{(i)}}{n} = T_t & , \text{ otherwise} \end{cases}$$

$$T_{k^*}(\underline{x}) = \begin{cases} T_{o,n} & , \text{ if } x_{(n)} < C\bar{x}, \quad C > 0 \\ T_t & , \text{ otherwise} \end{cases}$$

and showed $T_{k^*}(\underline{x})$ is preferred over $T_{o,n}$, T_t and D_{k^*} since the premium is small compared to the protection it provides.

Chikkagoudar and Kunchur (1980) proposed using $\hat{\theta}_{CK} = \sum_{i=1}^{n} w_i \hat{\theta}_i$

where $\hat{\theta}_i = \dfrac{\sum_{\substack{j=1 \\ j \neq i}}^{n} x_j}{n}$, $w_i = \dfrac{2r_i}{n(n+1)}$, r_i = rank of x_i in the complete sample. This estimator is more efficient than:

	TABLE 2.2.1	Alternative Estimator	
	$T_{o,n}$	$T_{m,n}$	$\hat{\theta}_{1,n}$
When to use $\hat{\theta}_{CK}$	$n \geq 4, k^* \leq .75$	$n \geq 6,\ .40 \leq k^* \leq .75$	all n, $k^* \geq .45$
	$n=3, k^* \leq .70$	$n=4,5,\ .35 \leq k^* \leq .75$	
	$n=2, k^* \leq .65$	$n=3,\ .30 \leq k^* \leq .70$	
		$n=2,\ .35 \leq k^* \leq .65$	

and it is independent of k^*.

2.2.2 Estimation for the normal distribution

Kale (1974a) has shown that the method of maximum likelihood applied to the exchangeable model involving normal distributions with change in location leads to estimators that are trimmed means.

For the case of $n-1$ observations from $N(\tau, \theta^2)$ and one observation from $N(\tau + k^*\theta, \theta^2)$, $k^* \geq 0$, $\bar{x}$ is UMVUE, MLE for τ if $k^* = 0$ but $\bar{x}$ is biased and $MSE(\bar{x}) = \dfrac{\theta^2 k^{*2}}{n}$ if $k^* > 0$. Thus an alternative estimator might be $T(\underline{x})$ where $T(\underline{x})$ is one of the following:

$$T(\underline{x}) = \begin{cases} \dfrac{\sum_{i=1}^{n-1} x_{(i)}}{n} = T_t \\ \dfrac{1}{n}\left\{\sum_{i=1}^{n-1} x_{(i)} + x_{(n-1)}\right\} = T_{1,n} \\ A(\underline{x}) = \begin{cases} \bar{x} & , \text{ if } x_{(n)} - \bar{x} \leq C_\alpha S \\ \dfrac{\sum_{i=1}^{n-1} x_{(i)}}{n-1} & , \text{ otherwise} \end{cases} \end{cases} .$$

The premium paid for using T instead of $\bar{x}$ is given by

$$\frac{1}{\theta^2} \mathrm{MSE}(T|k^*=0) - \frac{1}{\theta^2} \mathrm{MSE}(\bar{x}|k^*=0).$$

The protection obtained by using T instead of $\bar{x}$ is given by

$$\frac{1}{\theta^2} \mathrm{MSE}(\bar{x}|k^*>0) - \frac{1}{\theta^2} \mathrm{MSE}(T|k^*>0).$$

A detailed study of the accommodation of outliers in slippage models appears in Guttman and Smith (1969, 1971) and Guttman (1973a), using premium-protection. They consider three methods for estimating the mean $\mu = \tau$:

i) modified trimming $\hat{\mu}_A$ (A-rule)

ii) modified winsorization $\hat{\mu}_W$ (W-rule)

iii) semi-winsorization $\hat{\mu}_S$ (S-rule)

(see Appendix I).

Table 2.2.2. Comparison of $\hat{\mu}_A, \hat{\mu}_S$ and $\hat{\mu}_W$

Form of H_1 (one or two outliers)

	Location-slippage $N(\tau+k^*, \theta^2), k^*>0$	Scale-slippage $N(\tau, \theta^2 k^*), k^*>1$
$\hat{\mu}_A$	Best for large k^*	Not a contender
$\hat{\mu}_W$	Best for intermediate k^*	Best for large k^*
$\hat{\mu}_S$	Best for small k^*	Best for small k^*

The results are based on sample sizes up to $n = 20$, $\theta^2 = \sigma^2$ known, and only for $n = 3$, when θ^2 is unknown.

Guttman and Smith (1971) defined dispersion estimators $\hat{\sigma}_A^2$, $\hat{\sigma}_W^2$ and $\hat{\sigma}_S^2$ of $\hat{\sigma}^2$ analogously to $\hat{\mu}_A$, $\hat{\mu}_W$ and $\hat{\mu}_S$ (see Appendix I); $\hat{\sigma}_W^2$ is not worth considering when $\mu = \tau$ is known, $\hat{\sigma}_W^2$ and $\hat{\sigma}_A^2$ are not worth considering when τ is unknown.

CHAPTER III

The Gamma Distribution

One of the most common life-testing distributions is the gamma distribution. We shall examine the exchangeable model based on the gamma distribution and show that, in the case of a shape change, it is outlier-prone completely. For changes in shape or scale parameter, we shall determine which observation is most likely to be the spurious one. We shall also consider estimation in the presence of an outlier.

3.1 Characteristics of the Gamma Distribution

Consider the three-parameter gamma distribution given by

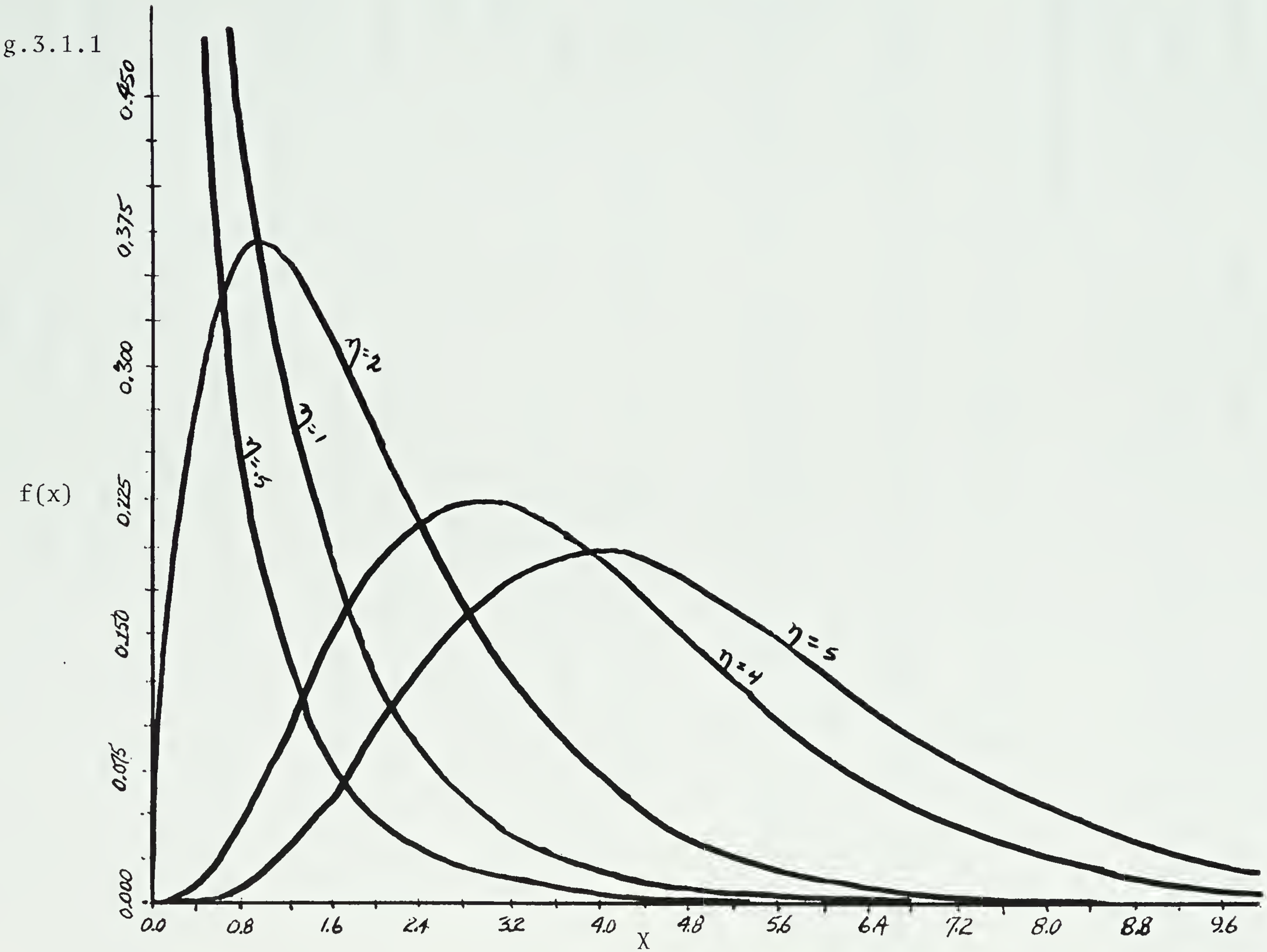
$$f(x;\theta,\eta,\tau) = \frac{(x-\tau)^{\eta-1} e^{-\frac{x-\tau}{\theta}}}{\theta^{\eta}\Gamma(\eta)}, \quad x > \tau,\ \theta,\eta > 0,\ -\infty < \tau < \infty.$$

We may denote this p.d.f. by GAM(θ,η,τ). The shape parameter is η, the scale parameter θ, and the location parameter τ. For $\tau = 0$ or known, this reduces to the two-parameter gamma distribution where

$$f(x;\theta,\eta) = \frac{e^{-\frac{x}{\theta}} x^{\eta-1}}{\theta^{\eta}\Gamma(\eta)}, \quad x > 0,\ \eta,\theta > 0.$$

For $\eta = 1$ we have the exponential distribution with parameter θ and for integer η we have the Erlang distribution. For $\eta \leq 1$ and fixed θ, the p.d.f. is decreasing in x and unbounded $(\eta<1)$ near the origin. The mean is $\theta\eta$, variance is $\theta^2\eta$, $E(X^r) = \frac{\theta^r\Gamma(\eta+r)}{\Gamma(\eta)}$ and the moment-generating function is $M_x(t) = (1-\theta t)^{-\eta}$, $t < \frac{1}{\theta}$. The shape parameter η is the reciprocal of the squared coefficient of variation.

Fig.3.1.1



Gamma p.d.f. for various values of η (θ =1, τ =0)

If T denotes the waiting time until the η^{th} occurrence in a Poisson process with intensity λ, then $T \sim GAM(\theta = \frac{1}{\lambda}, \eta, 0)$. The waiting time until the first occurrence may be denoted as $GAM(\theta = \frac{1}{\lambda}, 1, 0)$ which is $EXP(\theta = \frac{1}{\lambda})$. Thus the gamma distribution is a natural extension of the exponential distribution. The two-parameter exponential may be denoted as $GAM(\theta = \frac{1}{\lambda}, \eta = 1, \tau)$ where τ is a location parameter.

If $X_1, \ldots, X_n$ i.i.d. $GAM(\theta, m_i, 0)$ then $Y = \sum_{i=1}^{n} X_i \sim GAM(\theta, \sum_{i=1}^{n} m_i, 0)$. Consider the case of a component with $n-1$ spare parts. If X_i, $i = 1, 2, \ldots, n$ denote the lifetimes of the component and the spares and if each is distributed $GAM(\theta, 1, 0) = EXP(\theta = \frac{1}{\lambda})$, then the lifetime of the system, assuming use of the $n-1$ spares, is $Y = \sum_{i=1}^{n} X_i \sim GAM(\theta, n, 0)$ or $\frac{2Y}{\theta} \sim \chi^2_{2n\,df}$. Note also that if $Y \sim GAM(2, \eta, 0)$ then $Y \sim \chi^2_{2\eta\ df}$. The hazard function (HF) $h(x) = \lambda = \frac{1}{\theta}$ for $\eta = 1$; $h(x) \to \lambda^-$ as $x \to \infty$ for $\eta < 1$ and $h(0) = 0$; $h(x) \to \lambda^+$ as $x \to \infty$ and $h(0) = \infty$ for $\eta > 1$. Consequently the gamma distribution can model systems in a regular maintenance program where the failure rate is likely to increase initially but then stabilize.

Outliers in gamma samples arise in any context where Poisson processes are appropriate basic models, e.g. traffic flow, biological aggregation, failure of electronic equipment. They also occur in the context of a shifted exponential or gamma distribution. Outliers in χ^2 samples arise in ANOVA; outliers in gamma samples of arbitrary shape parameter arise with skew-distributed data, for which the gamma distribution is often a useful model.

3.2 Outlier-proneness of the exchangeable model with the Gamma distribution.

Neyman and Scott (1971) showed that for i.i.d.r.v.'s the family of gamma distributions indexed by shape parameter η is outlier-prone completely on the right. Let us consider now the exchangeable model.

CASE I: Scale change

Kale (1975b) showed that the exchangeable model with (at most) one spurious observation for scale parameter families for non-negative random variables involving a possible change in scale is outlier-prone completely. This would apply to the exchangeable model involving the gamma distribution with possible change in scale parameter θ.

CASE II: Shape change

Now consider the exchangeable model with (at most) one spurious observation based on the gamma distribution with possible change in the shape parameter η. Without loss of generality (w.l.o.g) we may take $\theta = 1$, $\tau = 0$.

Then $n-1$ observations have p.d.f. $f(x;\underline{\theta})$ which is $GAM(1,\eta,0)$ and one observation has p.d.f. $f(x;\underline{\xi})$ which is $GAM(1,k^*\eta,0)$ where $k^* \geq 1$. We shall show that this model is outlier-prone completely on the right.

Now we may write the likelihood as

$$L(x_1,\ldots,x_n;\eta,k^*) = \frac{1}{n}\sum_{r=1}^{n}\prod_{i\neq r} f(x_i;\eta)f(x_r;k^*\eta), \quad x_i \geq 0,\ k^* \geq 1,\ \eta > 0$$

($k^* \geq 1$ since we are considering $x_{(n)}$ as a possible outlier). Letting

E denote the event that $x_{(n)}$ is a (k,n)-outlier on the right and $P(k,n|L) = P(E|\eta,k^*)$, we must show that $\sup_{\substack{\eta>0 \\ k^*\geq 1}} P(E|\eta,k^*) = 1$ in order for this family to be (k,n)-outlier-prone.

The joint p.d.f. of the order statistics $x_{(1)},\ldots,x_{(n)}$ is given by

$$g(x_{(1)},\ldots,x_{(n)};\eta,k^*) = n!L(x_{(1)},\ldots,x_{(n)};\eta,k^*)$$

$$= (n-1)! \sum_{r=1}^{n} \prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta),$$

$$0 < x_{(1)} < \ldots < x_{(n)} < \infty .$$

Thus

$$P(E|\eta,k^*) = \int_E g(x_{(1)},\ldots,x_{(n)};\eta,k^*)dx_{(1)}\ldots dx_{(n)}$$

$$= \sum_{r=1}^{n} \int_E (n-1)! \prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta)dx_{(1)} \cdots dx_{(n)}$$

$$= \sum_{r=1}^{n} I_r(\eta,k^*)$$

and E is such that $0 < x_{(1)} < x_{(2)} < \ldots < x_{(n-1)} < \frac{kx_{(1)}+x_{(n)}}{k+1}$ $< x_{(n)} < \infty$.

Lemma 3.2.1: $0 \leq I_r(\eta,k^*) \leq u(r;n,\eta,k^*)$ where $u(r;n,\eta,k^*) = P\{X_{(r)}$ is the "spurious" observation whose p.d.f. is $f(x,k^*\eta)\}$.

Proof:

$$u(r;n,\eta,k^*) = (n-1)!\int_S \prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta)dx_{(1)},\dots,dx_{(n)} \text{ where}$$

S is such that $0 < x_{(1)} < x_{(2)} < \dots < x_{(n-1)} < x_{(n)} < \infty$.

Now $$I_r(\eta,k^*) = (n-1)!\int_E \prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta)dx_{(1)},\dots dx_{(n)} \text{ where}$$

E is such that $0 < x_{(1)} < x_{(2)} < \dots < x_{(n-1)} < \dfrac{kx_{(1)}+x_{(n)}}{k+1} < x_{(n)} < \infty$.

Since $E \subset S$, $0 \leq I_r(\eta,k^*) \leq u(r;n,\eta,k^*)$, $r = 1,2,\dots,n$.

Theorem 3.2.2: $$\lim_{k^*\to\infty} u(r;n,\eta,k^*) = \begin{cases} 1, & r = n \\ 0, & r = 1,2,\dots,n-1 \end{cases}.$$

Proof: Now $$u(r;n,\eta,k^*) = \binom{n-1}{r-1}\int_0^\infty \{F(y;\eta)\}^{r-1}\{1-F(y;\eta)\}^{n-r}f(y;k^*\eta)dy$$

where $F(y;\eta) = \int_{-\infty}^{y} f(x;\eta)dx$.

Consider now

$$u(n;n,\eta,k^*) = \int_{y=0}^{\infty}\left\{\int_{x=0}^{y} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}dx\right\}^{n-1} \frac{e^{-y}y^{k^*\eta-1}}{\Gamma(k^*\eta)}dy, \quad k^* \geq 1, \ \eta > 0 .$$

(without loss of generality $\theta = 1$, $\tau = 0$)

Let

$$I_n'(\eta,k^*) = \int_{y=0}^{\infty} \{\int_0^y \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\}^{n-1} e^{-y}y^{\nu}dy \quad \text{for} \quad \nu = k^*\eta-1.$$

Now setting $y = \nu t$, $dy = \nu dt$

$$I_n'(\eta,k^*) = \int_{t=0}^{\infty} \{\int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\}^{n-1} e^{-\nu t}(\nu t)^{\nu}\nu dt$$

$$= \nu^{\nu+1} \int_{t=0}^{\infty} \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

where

$$\vartheta(\nu,t) = [\int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx]^{n-1}.$$

and where there is no loss in generality in assuming $\nu > 0$ (i.e. $k^*\eta > 1$) since we are interested in $\lim_{k^*\to\infty} I_n'(\eta,k^*)$.

For $0 \le t \le \alpha$, $\vartheta(\nu,0) \le \vartheta(\nu,t) \le \vartheta(\nu,\alpha)$ and $0 \le \vartheta(\nu,t) \le 1$ for all ν,t and $\vartheta(\nu,t) \to 1$ a.e. in t as $\nu \to \infty$ (i.e. as $k^* \to \infty$). We may write $\int_0^{\infty} \vartheta(\nu,t)e^{\nu(\ln t-t)}dt = \int_0^1 \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$

$$+ \int_1^{\infty} \vartheta(\nu,t)e^{\nu(\ln t-t)}dt.$$

For each of the integrals on the right hand side, the dominant part occurs in the neighborhood of the point when $\ln t-t$ is maximum, i.e. $t = 1$. Following Copson (1967, p. 36 ff), since $0 \le \vartheta(\nu,t) \le 1$ for all ν and t and $\vartheta(\nu,t)$ is continuous and increasing in t there

is a number $t_0 (0<t_0\le 1)$ such that

$$\int_0^1 \vartheta(\nu,t)e^{\nu(\ln t-t)}dt = \vartheta(\nu,t_0)\int_0^1 e^{\nu(\ln t-t)}dt\ , \quad 0\le t_0\le 1\ .$$

We cannot have $t_0 = 0$ since $\vartheta(\nu,0) = 0$ while

$$\int_0^1 \vartheta(\nu,t)e^{\nu(\ln t-t)}dt > \int_{.5}^1 \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

$$> \frac{1}{2}\vartheta(\nu,\ \frac{1}{2})e^{-\nu(\frac{1}{2}-\ln 2)} > 0$$

and

$$\int_1^\infty \vartheta(\nu,t)e^{\nu(\ln t-t)}dt = \lim_{\alpha\to\infty}\int_1^\alpha \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

$$= \lim_{\alpha\to\infty}\vartheta(\nu,t_1)\int_1^\alpha e^{\nu(\ln t-t)}dt,\quad 1\le t_1\le \alpha.$$

Thus

$$\int_0^\infty \vartheta(\nu,t)e^{\nu(\ln t-t)}dt \sim \int_0^\infty e^{\nu(\ln t-t)}dt$$

$$\sim 1\cdot\frac{\Gamma(\nu+1)}{\nu^{\nu+1}} \quad \text{as} \quad \nu\to\infty$$

(see Appendix II). As a result, we may write

$$I_n'(\eta,k^*) \sim \nu^{\nu+1}\cdot 1\cdot \frac{\Gamma(\nu+1)}{\nu^{\nu+1}} = \Gamma(\nu+1) \quad \text{as} \quad \nu \to \infty$$

and

$$u(n;n,\eta,k^*) = \frac{I_n'(\eta,k^*)}{\Gamma(k^*\eta)} \quad \text{where} \quad \nu = k^*\eta-1$$

$$\to 1 \quad \text{as} \quad \nu \to \infty \quad \text{(i.e. as } k^* \to \infty).$$

For $u(r;n,\eta,k^*)$, $r = 1,2,\ldots,n-1$ we have

$$0 \leq u(r;n,\eta,k^*) = \binom{n-1}{r-1}\int_{y=0}^{\infty} \left\{\int_0^y \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{r-1}$$

$$\left\{1 - \int_0^y \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{n-r} \frac{e^{-y}y^{k^*\eta-1}}{\Gamma(k^*\eta)}\,dy$$

$$= \binom{n-1}{r-1}\frac{I_r'(\eta,k^*)}{\Gamma(k^*\eta)}$$

where

$$I_r'(\eta,k^*) = \int_{t=0}^{\infty} \left\{\int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{r-1}\left\{1 - \int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{n-r} e^{-\nu t}\nu^{\nu+1}t^{\nu}\,dt,$$

$\nu = k^*\eta-1$, $y = \nu t$, $dy = \nu dt$ and again we assume $\nu > 0$. Therefore

$$I'_r(\eta,k^*) = \nu^{\nu+1}\int_0^\infty \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

$$= \nu^{\nu+1}\lim_{\alpha\to\infty}\int_0^\alpha \vartheta(\nu,t)e^{\nu(\ln t-t)}dt .$$

Now $\vartheta(\nu,t) = \{\int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}dx\}^{n-1}\{1 - \int_0^{\nu t}\frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}dx\}^{n-r} \leq 1$ and as $k^* \to \infty$, $\nu \to \infty$ and $\vartheta(\nu,t) \to 0$ almost everywhere on $[0,\infty]$. Since $\vartheta(\nu,t)$ is continuous and bounded for all t

$$\int_0^\alpha \vartheta(\nu,t)e^{\nu(\ln t-t)}dt = \vartheta(\nu,t_0)\int_0^\alpha e^{\nu(\ln t-t)}dt \qquad 0 \leq t_0 \leq \alpha$$

$$\leq \vartheta(\nu,t_0)\int_0^\infty e^{\nu(\ln t-t)}dt = \vartheta(\nu,t_0)\frac{\Gamma(\nu+1)}{\nu^{\nu+1}} .$$

Therefore $0 \leq u(r;n,\eta,k^*) \leq \frac{\binom{n-1}{r-1}}{\Gamma(k^*\eta)}\nu^{\nu+1}\lim_{\alpha\to\infty}\frac{\vartheta(\nu,t_0)\Gamma(\nu+1)}{\nu^{\nu+1}} = \binom{n-1}{r-1}$ $\lim_{\alpha\to\infty}\vartheta(\nu,t_0)$. But $\binom{n-1}{r-1}\lim_{\alpha\to\infty}\vartheta(\nu,t_0) \to 0$ as $\nu \to \infty$ (i.e. as $k^* \to \infty$) and hence $u(r;n,\eta,k^*) \to 0$ as $\nu \to \infty$ for $r = 1,2,\ldots,n-1$.

Theorem 3.2.3: The exchangeable model with (at most) one spurious observation based on the gamma distribution with a change in shape parameter is outlier-prone completely on the right.

Proof: We need to show $\sup_{\substack{\eta>0\\ k^*\geq 1}} P(E|\eta,k^*) = 1$.

Since $P(E|\eta,k^*) = \sum_{r=1}^{n} I_r(\eta,k^*)$,

$$\lim_{k^*\to\infty} P(E|\eta,k^*) = \sum_{i=1}^{n} \lim_{k^*\to\infty} I_r(\eta,k^*).$$

From Lemma 3.2.1, $0 \leq I_r(\eta,k^*) \leq u(r;n,\eta,k^*)$, $r = 1,2,\ldots,n$.

By Theorem 3.2.2, $\lim_{k^*\to\infty} u(r;n,\eta,k^*) = \begin{cases} 0 & , r = 1,2,\ldots,n-1 \\ 1 & , r = n \end{cases}$.

Thus $\lim_{k^*\to\infty} I_r(\eta,k^*) = 0, \quad r = 1,2,\ldots,n-1$.

and $\lim_{k^*\to\infty} P(E|\eta,k^*) = \lim_{k^*\to\infty} I_n(\eta,k^*)$.

Now we need only show that $\lim_{k^*\to\infty} I_n(\eta,k^*) = 1$ in order for this model to be outlier-prone completely on the right.

But

$$I_n(\eta,k^*) = (n-1)!\int_E \prod_{i\neq n} f(x_{(i)};\eta)f(x_{(n)};k^*\eta)dx_{(1)} \cdots dx_{(n)}$$

$$= (n-1)!\int_E \prod_{i\neq n} \frac{e^{-x_{(i)}}x_{(i)}^{\eta-1}}{\Gamma(\eta)} \frac{e^{-x_{(n)}}x_{(n)}^{k^*\eta-1}}{\Gamma(k^*\eta)} dx_{(1)} \cdots dx_{(n)}$$

where E is such that $0 < x_{(1)} < x_{(2)} < \cdots < x_{(n-1)} < \frac{kx_{(1)}+x_{(n)}}{k+1} < x_{(n)} < \infty$.

Let $y_1,\ldots,y_n$ denote the order statistics $x_{(1)},\ldots,x_{(n)}$, respectively. Then we may write

$$I_n(\eta,k^*) = (n-1) \iint_{0<y_1<y_n<\infty} \left\{\int_0^{\frac{ky_1+y_n}{k+1}} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx - \int_0^{y_1} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{n-2}$$

$$\frac{e^{-y_1}y_1^{\eta-1}}{\Gamma(\eta)} \frac{e^{-y_n}y_n^{k^*\eta-1}}{\Gamma(k^*\eta)}\,dy_1 dy_n$$

$$= \int_{y_n=0}^{\infty} \left[\int_{y_1=0}^{y_n} (n-1)\left\{F\left(\frac{ky_1+y_n}{k+1};\eta\right)-F(y_1;\eta)\right\}^{n-2} f(y_1;\eta)dy_1\right]$$

$$f(y_n;k^*\eta)dy_n$$

where $f(y;\eta) = \dfrac{e^{-y}y^{\eta-1}}{\Gamma(\eta)}$, $y > 0$. Then

$I_n(\eta,k^*) = \dfrac{1}{\Gamma(k^*\eta)} \displaystyle\int_{y_n=0}^{\infty} \vartheta(y_n)e^{-y_n}y_n^{k^*\eta-1}dy_n$ where

$\vartheta(y_n) = \displaystyle\int_{y_1=0}^{y_n} (n-1)\left\{F\left(\frac{ky_1+y_n}{k+1};\eta\right)-F(y_1;\eta)\right\}^{n-2} f(y_1;\eta)dy_1$. Setting

$y_n = \nu t$ where $\nu = k^*\eta-1$, we obtain

(3.2.1) $$I_n(\eta,k^*) = \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \int_{t=0}^{\infty} \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

$$= \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \lim_{\alpha\to\infty} \int_{t=0}^{\alpha} \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

where $\vartheta(\nu,t) = \vartheta(\nu t) = \displaystyle\int_{y_1=0}^{\nu t} (n-1)\left\{F\left(\frac{ky_1+\nu t}{k+1};\eta\right)-F(y_1;\eta)\right\}^{n-2} f(y_1;\eta)dy_1$.

As $k^* \to \infty$, $\nu \to \infty$ and the major contribution to this integral in 3.2.1 occurs in the neighborhood of $t = 1$. Also $0 \leq \vartheta(\nu,t) \leq 1$ for all ν,t and $\vartheta(\nu,t) \to 1$ a.e. on $[0,\infty)$ as $\nu \to \infty$ (i.e. as $k^* \to \infty$), since

$$\lim_{\nu\to\infty} \left[F\left(\frac{ky_1+\nu t}{k+1}\,;\eta\right)-F(y_1;\eta)\right] = 1-F(y_1;\eta) \text{ a.e. on } [0,\infty).$$

If $t = 0$, $\vartheta(\nu,t) = (n-1)\int_0^{\nu t} \left[F\left(\frac{ky_1+\nu t}{k+1}\,;\eta\right)-F(y_1;\eta)\right]^{n-2} f(y_1;\eta)dy_1$ and $\vartheta(\nu,0) = 0$. But $\vartheta(\nu,t)$ is nonnegative, bounded, increasing and absolutely continuous in t. Thus $\exists t_0 \in [0,\infty) \ni .\ \int_0^{\alpha} \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$ $= \vartheta(\nu,t_0)\int_0^{\alpha} e^{\nu(\ln t-t)}dt$. On the other hand for any $0 < \xi < 1$

$$\int_0^{\alpha} \vartheta(\nu,t)e^{\nu(\ln t-t)}dt > \int_{\xi}^{1} \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

$$> \vartheta(\nu,\xi)e^{\nu(\ln \xi-\xi)}(1-\xi)$$

$$> 0.$$

Therefore $t_0 \neq 0$.

Thus $\int_0^{\alpha} \vartheta(\nu,t)e^{\nu(\ln t-t)}dt \sim \int_0^{\infty} e^{\nu(\ln t-t)}dt = \frac{\Gamma(\nu+1)}{\nu^{\nu+1}}$ as $\nu \to \infty$

and

$$I_n(\eta,k^*) = \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \lim_{\alpha\to\infty} \int_0^{\alpha} \vartheta(\nu,t) e^{\nu(\ell n\ t-t)} dt$$

$$\sim \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \cdot \frac{\Gamma(\nu+1)}{\nu^{\nu+1}} \quad \text{as} \quad \nu \to \infty \quad (\text{i.e. as } k^* \to \infty).$$

Since $\lim_{k^*\to\infty} I_n(\eta,k^*) = 1$ and $\lim_{k^*\to\infty} I_r(\eta,k^*) = 0$, $r = 1,2,\ldots,n-1$ and $P(E|\eta,k^*) = \sum_{r=1}^{n} I_r(\eta,k^*)$,

$$\sup_{\substack{\eta>0 \\ k^*\geq 1}} P(E|\eta,k^*) = 1$$

and the family is outlier-prone completely on the right.

Thus for either a scale or shape change, the exchangeable model with at most one spurious observation based on the gamma distribution is outlier-prone completely on the right.

3.3 Detection of Outliers

Mount and Kale (1973) considered a general model, assuming $X_1,\ldots,X_n$ are such that $n-1$ of them are distributed with distribution function (d.f) $F(x)$ and one is distributed with d.f. $G(x)$, where F and G are stochastically ordered, i.e. $G < F$. A priori, each X_i has probability $\frac{1}{n}$ of being the spurious observation distributed as G. Let $\Psi(x) = \frac{dG}{dF}$. If $\Psi(x)$ is monotone increasing, then $u(1;n,k^*) < u(2;n,k^*) < \ldots < u(n;n,k^*)$ where $u(i;n,k^*) = P(X_{(i)}$ is the spurious observation in a sample of size n with k^* the coefficient of spuriosity).

CASE I: Scale change: Consider a situation where $n-1$ observations are from $GAM(\theta,\eta,0)$ and one is from $GAM(k^*\theta,\eta,0)$, $k^* > 0$. If $k^* = 1$ we have homogeneous data. Now

$$\Psi(x) = \frac{dG(x)}{dF(x)} = \frac{e^{-\frac{x}{\theta}\left(\frac{1-k^*}{k^*}\right)}}{k^{*\eta}}$$

and

$$\Psi'(x) = \frac{e^{-\frac{x}{\theta}\left(\frac{1-k^*}{k^*}\right)}}{k^{*\eta}} \left(\frac{k^*-1}{\theta^{k^*}}\right) \begin{cases} > 0 & \text{if } k^* > 1 \\ < 0 & \text{if } k^* < 1 \end{cases}.$$

Thus, if $k^* > 1$, $\Psi(x)$ is monotone increasing and $X_{(n)}$ has maximum probability of being spurious; if $k^* < 1$, Ψ is monotone decreasing and $X_{(1)}$ has maximum probability of being spurious.

<u>CASE II: Shape change</u>: If $n-1$ observations come from $GAM(\theta,\eta,0)$ and one is from $GAM(\theta,k^*\eta,0)$, $k^* > 0$, assuming the exchangeable model,

$$\Psi(x) = \frac{dG}{dF} = \frac{x^{k^*\eta-\eta}\Gamma(\eta)}{\theta^{k^*\eta-\eta}\Gamma(k^*\eta)}$$

and

$$\Psi'(x) = \frac{\Gamma(\eta)\eta(k^*-1)x^{k^*\eta-\eta-1}}{\Gamma(k^*\eta)\theta^{k^*\eta-\eta}} \begin{cases} > 0 & \text{if } k^* > 1 \\ < 0 & \text{if } k^* < 1 \ . \end{cases}$$

Thus, if $k^* > 1$, $\Psi(x)$ is monotone increasing and $X_{(n)}$ has maximum probability of being spurious; for $0 < k^* < 1$, $\Psi(x)$ and $u(r;n,k^*)$ are monotone decreasing and $X_{(1)}$ has maximum probabilty of being spurious.

This now shows that the spurious observation resulting from a scale or shape change is most likely to occur at the sample extremes i.e. it tends to show up as an outlier.

3.4 Estimation for Gamma parameters

3.4.1 Standard Estimators

The gamma distribution is a member of the exponential class, consequently $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} \ln X_i\right)$ are complete sufficient statistics for (θ, η).

For known shape η, the MLE $\hat{\lambda}_{MLE}$ of $\lambda = \frac{1}{\theta}$ is $\frac{\eta}{\bar{x}}$ which is biased but consistent and asymptotically $N\left(\lambda, \frac{\lambda^2}{n\eta}\right)$. Thus the UMVUE for θ (η known) is $\frac{\bar{x}}{\eta}$.

For unknown η, we obtain

$$\hat{\theta}_{MLE} = \frac{\bar{x}}{\hat{\eta}}$$

$$g(\hat{\eta}_{MLE}) = \ln \hat{\eta}_{MLE} - \psi(\hat{\eta}_{MLE}) - \ln \bar{x} + \ln \tilde{x} = 0$$

where $\tilde{x} = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$ and $\psi(z)$ is Euler's psi function i.e.

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \int_0^\infty \frac{\frac{e^{-t}}{t} - e^{-zt}}{1-e^{-t}} dt = -\gamma + \frac{1}{z} + z \sum_{i=1}^{\infty} [i(i+z)]^{-1} \quad \text{where}$$

$\gamma = .5772157$ (Euler's constant). These equations must be solved iteratively (see Choi and Wette (1969)). For large $\hat{\eta}_{MLE}$, Linhart (1965) suggested approximating $\ln \hat{\eta}_{MLE} - \psi(\hat{\eta}_{MLE})$ by $(2\hat{\eta}_{MLE} - 1/3)^{-1}$ thus

$$\hat{\eta}_{MLE} = \{(\ln \bar{x} - \ln \tilde{x})^{-1} + 1/3\}1/2.$$

Moment estimators give

$$\hat{\eta}_M = \frac{(\sum_{i=1}^{n} X_i)^2}{\sum_{i=1}^{n} n(X_i - \bar{X})^2}$$

$$\hat{\theta}_M = \frac{\sum_{i=1}^{n} X_i}{n\eta_M} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\sum_{i=1}^{n} X_i} .$$

These have lower asymptotic efficiency than MLE's but Lilliefors (1971) showed that for $n \leq 20$, $\eta \geq 2$, the MSE's of the moment estimators are close to those of the MLE's. Both the MLE's and the moment estimators are biased. To approximate zero bias and smaller MSE, Lilliefors suggested the following corrections:

TABLE 3.4.1 Lilliefors' improved estimators of η and $\lambda = 1/\theta$

	Based on M.L.E.	Based on Moment Estimators
$\hat{\hat{\eta}}$	$\frac{\hat{\eta}_{MLE}}{1+3/n}$	$\frac{\hat{\eta}_M}{1+2/n} - 5/3$
$\hat{\hat{\lambda}} = \frac{\hat{\hat{1}}}{\theta}$	$\frac{n\hat{\eta}_{MLE}}{(1+\frac{3}{n})\sum_{i=1}^{n} X_i}$	$\{\frac{n\hat{\eta}_M}{(1+2/n)} - 5/3\} \frac{1}{\sum_{i=1}^{n} X_i}$

Thom (1968) has given estimators very similar to the MLE's for $\eta > 1$.

$$\eta^* = \frac{1 + \sqrt{1 + \frac{4M}{3}}}{4M}$$

$$\theta^* = \frac{\bar{x}}{\eta^*}$$

where $M = \ln(\bar{x}/\tilde{x}) = -\ln S_1$ and S_1 is sufficient for η and independent of θ. Bain and Englehardt (1975) have a chi-square approximation for M valid for all η (θ acts as a nuisance parameter).

$$2n\eta Mc \sim \chi^2_{\nu df}$$

where $W = 2n\eta M$, $c = 2\,\frac{E(W)}{Var(W)} = \frac{nw_1(\eta)-w_1(n\eta)}{nw_2(\eta)-w_2(n\eta)}$ and $\nu = 2\,\frac{[E(W)]^2}{Var(W)} = [nw_1(\eta)-w_1(n\eta)]c$ and $w_1(z) = 2z\{\ln z-\psi(z)\}$, $w_2(z) = 2z\{z\psi'(z)-1\}$ where $\psi(z)$ is the psi function. As $\eta \to 0$, $W \xrightarrow{d} \chi^2_{2(n-1)df}$.

$$w_1(\eta) \doteq 1 + \frac{1}{1+6\eta}$$

$$w_2(\eta) \doteq \begin{cases} 1 + \frac{1}{1+2.5\eta}, & 0 < \eta < 2 \\ 1 + \frac{1}{3\eta}, & \eta \geq 2 \end{cases}$$

$$\frac{\nu}{n-1} \doteq 1 + \frac{1}{(1+4.3\eta)^2}$$

3.4.2 Estimators suggested for use

Case I: Scale Change

Assuming the exchangeable model where $n-1$ observations are distributed as $GAM(\theta,\eta,0)$ and one is distributed as $GAM(\theta k^*,\eta,0)$, η known, $k^* \geq 1$, we have for $k^* = 1$, homogeneous data and the best linear unbiased estimator (BLUE) for θ is $\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{n\eta}$ and $E(\hat{\theta}_{MLE}) = \theta$ and $MSE(\hat{\theta}_{MLE}) = Var(\hat{\theta}_{MLE}) + \{Bias(\hat{\theta}_{MLE})\}^2 = \frac{\theta^2}{n\eta}$. For the heterogeneous case $(k^*>1)$, $E_{het}(\hat{\theta}_{MLE}) = \frac{1}{n\eta}\sum_{i=1}^{n}\left(\frac{n-1}{n}\theta\eta + \frac{1}{n}k^*\theta\eta\right)$ $= \frac{\theta}{n}(n-1+k^*)$ and $Bias_{het}(\hat{\theta}_{MLE}) = \frac{\theta(k^*-1)}{n}$. As $k^* \to \infty$, this bias $\to \infty$. Now $MSE_{het}(\hat{\theta}_{MLE}) = Var_{het}(\hat{\theta}_{MLE}) + \{Bias_{het}(\hat{\theta}_{MLE})\}^2$ and

$$Var_{het}(\hat{\theta}_{MLE}) = \frac{1}{(n\eta)^2}\sum_{i=1}^{n}\left\{\frac{n-1}{n}\theta^2\eta + \frac{1}{n}(k^*\theta)^2\eta\right\}$$

$$= \frac{\theta^2}{n^2\eta}\{k^{*2}+n-1\}.$$

therefore

$$MSE_{het}(\hat{\theta}_{MLE}) = \frac{\theta^2}{n^2\eta}(k^{*2}+n-1) + \frac{\theta^2(k^*-1)^2}{n^2}$$

$$= \frac{\theta^2}{n^2\eta}\{k^{*2}+n-1+\eta(k^*-1)^2\}$$

As $k^* \to \infty$, $MSE_{het}(\hat{\theta}_{MLE}) \to \infty$ and hence $\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{n\eta}$ proves to be a poor estimator of θ.

If we consider a random sample of size n where $n-m$ observations are from GAM($\theta,\eta,0$) and m are from GAM($k^*\theta,\eta,0$), $k^* > 1$ (i.e. same shape but a change in scale parameter) and <u>a priori</u> each subset of m observations is equally likely to be the "outlier subset", then the likelihood

$$L(\underline{x}|\theta,\eta,k^*,I) = \frac{1}{\binom{n}{m}} \frac{e^{-\sum_{x_i \notin I} \frac{x_i}{\theta}} \prod_{x_i \notin I} x_i^{\eta-1}}{\{\Gamma(\eta)\theta^{\eta}\}^{n-m}} \frac{e^{-\sum_{x_i \in I} \frac{x_i}{k^*\theta}} \prod_{x_i \in I} x_i^{\eta-1}}{\{\Gamma(\eta)(k^*\theta)^{\eta}\}^{m}}$$

$$= \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} x_i^{\eta-1} e^{-\left[\sum_{x_i \notin I} \frac{x_i}{\theta} + \sum_{x_i \in I} \frac{x_i}{\theta_1}\right]}}{\{\Gamma(\eta)\}^{n} \theta^{\eta(n-m)} \theta_1^{\eta m}}$$

for $k^*\theta = \theta_1$ where $I = (x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ and $I \in \mathcal{T}$, the collection of all possible combinations of m observations out of n. (I represents the subset of m spurious observations). To maximize $L(\underline{x}|\theta,\eta,k^*,I)$ for $\theta,\eta > 0$, $I \in \mathcal{T}$, $k^* > 1$ we use the fact that $\Psi(x) = \frac{dG(x)}{dF(x)}$ is monotone increasing for $k^* > 1$ and hence $\max_{I \in \mathcal{T}} L(\underline{x}|\theta,\eta,k^*,I)$ occurs at $I = \hat{I} = (x_{(n-m+1)}, \ldots, x_{(n)})$. Therefore

$$L(\underline{x}|\theta,\eta,k^*,\hat{I}) = \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} x_i^{\eta-1} e^{-\left\{\sum_{i=1}^{n-m} \frac{x_{(i)}}{\theta} + \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{\theta_1}\right\}}}{\{\Gamma(\eta)\}^n \theta^{\eta(n-m)} \theta_1^{\eta m}}$$

and

$$K(\underline{x}|\theta,\eta,k^*,\hat{I}\} = \ell n\ L(\underline{x}|\theta,\eta,k^*,\hat{I})$$

$$= C + (\eta-1)\sum_{i=1}^{n} \ell n\ x_i - \sum_{i=1}^{n-m} \frac{x_{(i)}}{\theta} - \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{\theta_1}$$

$$- n\ell n\ \Gamma(\eta) - \eta(n-m)\ell n\ \theta - \eta m \ell n\ \theta_1 .$$

Assuming

i) θ, θ_1 and η are unknown, we obtain:

$$\frac{\partial K}{\partial \eta} = \sum_{i=1}^{n} \ell n\ x_i - \frac{n\Gamma'(\eta)}{\Gamma(\eta)} - \{(n-m)\ell n\ \theta + m\ell n\ \theta_1\}$$

$$\frac{\partial K}{\partial \theta} = \sum_{i=1}^{n-m} \frac{x_{(i)}}{\theta^2} - \frac{\eta(n-m)}{\theta}$$

$$\frac{\partial K}{\partial \theta_1} = \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{\theta_1^2} - \frac{\eta m}{\theta_1} .$$

Setting the above three equations equal to zero, we obtain

$$\hat{\theta} = \frac{\sum_{i=1}^{n-m} x_{(i)}}{\eta(n-m)}$$

$$\hat{\theta}_1 = \frac{\sum_{i=n-m+1}^{n} x_{(i)}}{\eta(n-m)}$$

$$h(\hat{\eta}) = \sum_{i=1}^{n} \ln x_{(i)} - \frac{n\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} + n\ln\hat{\eta}$$

$$- (n-m)\ln \frac{\sum_{i=1}^{n-m} x_{(i)}}{n-m} - m\ln \frac{\sum_{i=n-m+1}^{n} x_{(i)}}{m} = 0.$$

ii) for the case of known η, we obtain

$$\hat{\theta} = \frac{\sum_{i=1}^{n-m} x_{(i)}}{\eta(n-m)} \quad \text{and} \quad \hat{\theta}_1 = \frac{\sum_{i=n-m+1}^{n} x_{(i)}}{m\eta}$$

which are trimmed means.

iii) for the case of known θ, we may use $Y_i = \frac{X_i}{\theta}$.

$$L(\underline{y};\eta,k^*,I) = \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} y_i^{\eta-1} e^{-\{\sum_{i\notin I} y_i + \sum_{i\in I} y_i/k^*\}}}{\{\Gamma(\eta)\}^n k^{*\,\eta m}}$$

and

$$L(\underline{y};\eta,k^*,\hat{I}) = \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} y_i^{\eta-1} e^{-\{\sum_{i=1}^{n-m} y_{(i)} + \sum_{i=n-m+1}^{n} y_{(i)}/k^*\}}}{\{\Gamma(\eta)\}^n k^{*\,\eta m}}$$

and

$$K(\underline{y};\eta,k^*,\hat{I}) = \ln L(\underline{y};\eta,k^*,\hat{I})$$

$$= C' + (\eta-1)\sum_{i=1}^{n} \ln y_i - \{\sum_{i=1}^{n-m} y_{(i)} + \sum_{i=n-m+1}^{n} y_{(i)}/k^*\}$$

$$- n \ln \Gamma(\eta) - \eta m \ln k^*.$$

Then

$$\frac{\partial K}{\partial \eta} = \sum_{i=i}^{n} \ln y_i - \frac{n\Gamma'(\eta)}{\Gamma(\eta)} - m \ln k^*$$

$$\frac{\partial K}{\partial k^*} = \sum_{i=n-m+1}^{n} \frac{y_{(i)}}{k^{*2}} - \frac{m\eta}{k^*}$$

and thus

$$\hat{k}^* = \frac{\sum_{i=n-m+1}^{n} y_{(i)}}{m\eta}$$

and

$$\frac{n\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} - m \ln \hat{\eta} = \sum_{i=1}^{n} \ln y_i - m \ln \left\{ \frac{\sum_{i=n-m+1}^{n} y_{(i)}}{m} \right\} .$$

iv) for the case of known k*

$$\frac{\partial K}{\partial \eta} = \sum_{i=i}^{n} \ell n\ x_i - \frac{n\Gamma'(\eta)}{\Gamma(\eta)} - (n-m)\ell n\ \theta - m\ \ell n(k^*\theta)$$

$$\frac{\partial K}{\partial \theta} = \sum_{i=i}^{n-m} \frac{x_{(i)}}{\theta^2} + \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{k^*\theta^2} - \frac{\eta(n-m)}{\theta} - \frac{\eta m}{\theta}\ .$$

Setting $\frac{\partial K}{\partial \eta} = 0$ and $\frac{\partial K}{\partial \theta} = 0$, we obtain

$$\hat{\eta}\hat{\theta} = \frac{\sum_{i=1}^{n-m} x_{(i)} + \frac{1}{k^*} \sum_{i=n-m+1}^{n} x_{(i)}}{n}$$

$$\ell n\ \hat{\theta} + \frac{\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} = \frac{\sum_{i=1}^{n} \ell n\ x_i - m\ \ell n\ k^*}{n}\ .$$

Case II. Shape change

Assuming n-1 observations distributed as $GAM(\theta,\eta,0)$ and one distributed as $GAM(\theta,k^*\eta,0)$, $k^* \geq 1$, we have, for heterogeneous data ($k^*>1$),

$$E_{het}(\hat{\theta}_{MLE}) = E\left(\frac{\bar{X}}{\eta}\right) = \frac{\theta}{n\eta} \sum_{i=1}^{n} \left(\frac{n-1}{n}\ \theta\eta + \frac{1}{n}\ \theta k^*\eta\right)$$

$$= \theta + \frac{\theta(k^*-1)}{n}\ .$$

As $k^* \to \infty$, bias $\to \infty$. Also

$$\mathrm{Var}_{het}(\hat{\theta}_{MLE}) = \mathrm{Var}\left(\frac{\bar{X}}{\eta}\right) = \frac{1}{n^2\eta^2} \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

$$= \frac{1}{n^2\eta^2} \sum_{i=1}^{n} \left(\frac{n-1}{n}\,\theta^2\eta + \frac{1}{n}\,\theta^2 k^*\eta\right)$$

$$= \frac{\theta^2}{n^2\eta}\,(n+k^*-1).$$

Then

$$\mathrm{MSE}_{het}(\hat{\theta}_{MLE}) = \mathrm{Var}_{het}(\hat{\theta}_{MLE}) + \{\mathrm{Bias}\ (\hat{\theta}_{MLE})\}^2$$

$$= \frac{\theta^2(n+k^*-1)}{n^2\eta} + \frac{\theta^2\eta(k^*-1)^2}{\eta n^2}$$

$$= \frac{\theta^2\{n+k^*-1+\eta(k^*-1)^2\}}{n^2\eta}\ .$$

As $k^* \to \infty$, $\mathrm{MSE}_{het}(\hat{\theta}_{MLE}) \to \infty$ and hence $\hat{\theta} = \frac{\bar{x}}{\eta}$ proves to be a poor estimator of θ (We have assumed η known).

Consider now the exchangeable model where $n-m$ observations are from $GAM(\theta,\eta,0)$ and m observations are from $GAM(\theta,k^*\eta,0)$, $k^* \geq 1$ (i.e. possible change in shape; scale constant). Then

$$L(\underline{x}|\theta,\eta,k^*,I) = \frac{1}{\binom{n}{m}} \frac{e^{-\sum_{x_i \notin I} \frac{x_i}{\theta}} \prod_{x_i \notin I} x_i^{\eta-1}}{\{\Gamma(\eta)\}^{n-m}(\theta^\eta)^{n-m}} \frac{e^{-\sum_{x_i \in I} \frac{x_i}{\theta}} \prod_{x_i \in I} x_i^{k^*\eta-1}}{\{\Gamma(k^*\eta)\}^{n-m}(\theta^{k^*\eta})^m}$$

$$= \frac{1}{\binom{n}{m}} \frac{e^{-T/\theta} \prod_{x_i \notin I} x_i^{\eta-1} \prod_{x_i \in I} x_i^{k^*\eta-1}}{\{\Gamma(\eta)\}^{n-m}\{\Gamma(k^*\eta)\}^m(\theta^\eta)^{n-m+k^*m}}$$

where $T = \sum_{i=1}^{n} x_i$, $I = (x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ and $I \in \mathcal{T}$, the collection of all possible subsets of m observations out of n. To obtain MLE's, note that $\Psi(x) = \frac{dG}{dF}$ is monotone increasing and, by Kale (1975b), we know $\max_{I \in \mathcal{T}} L(\underline{x}|\theta,\eta,k^*,I)$ occurs at $\hat{I} = I(\theta,\eta,k^*$ fixed) where $\hat{I} = (x_{(n-m+1)},\ldots,x_{(n)})$. Thus

$$\max_{\substack{k^*>1 \\ I\in\mathcal{T}}} L(\underline{x}|\theta,\eta,k^*,I) = \max_{k^*\geq 1} L(\underline{x}|\theta,\eta,k^*,\hat{I})$$

and

$$L(\underline{x}|\theta,\eta,k^*,\hat{I}) = \frac{e^{-T/\theta} \prod_{i=1}^{n-m} x_{(i)}^{\eta-1} \prod_{i=n-m+1}^{n} x_{(i)}^{k^*\eta-1}}{\binom{n}{m} \{\Gamma(\eta)\}^{n-m}\{\Gamma(k^*\eta)\}^m(\theta^\eta)^{n-m+k^*m}} .$$

Using $\eta_1 = k^*\eta$ ($> \eta$ since $k^* > 1$)

$$K(\underline{x}|\theta,\eta,\eta_1,\hat{I}) = \ell n\ L(\underline{x}|\theta,\eta,\eta_1,\hat{I})$$

$$= C - \frac{T}{\theta} + (\eta-1) \sum_{i=1}^{n-m} \ell n\ x_{(i)} + (\eta_1-1) \sum_{i=n-m+1}^{n} \ell n\ x_{(i)}$$

$$- (n-m)\ln \Gamma(\eta) - m \ln \Gamma(\eta_1) - (n-m)\eta \ln \theta - m\eta_1 \ln \theta$$

and

i) assuming θ, η, and η_1 unknown, we obtain

$$\frac{\partial K}{\partial \theta} = \frac{T}{\theta^2} - \frac{(n-m)\eta}{\theta} - \frac{m\eta_1}{\theta}$$

$$\frac{\partial K}{\partial \eta} = \sum_{i=1}^{n-m} \ln x_{(i)} - \frac{(n-m)\Gamma'(\eta)}{\Gamma(\eta)} - (n-m) \ln \theta$$

$$\frac{\partial K}{\partial \eta_1} = \sum_{i=n-m+1}^{n} \ln x_{(i)} - \frac{m\Gamma'(\eta_1)}{\Gamma(\eta_1)} - m \ln \theta \quad .$$

Now $\frac{\partial K}{\partial \theta} = 0$ implies $\hat{\theta} = \frac{T}{(n-m)\hat{\eta} + m\hat{\eta}_1}$ and

$$\frac{\partial K}{\partial \eta} = 0 \text{ implies } \frac{\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} + \ln \hat{\theta} = \frac{\sum_{i=1}^{n-m} \ln x_{(i)}}{n-m} \text{ and}$$

$$\frac{\partial K}{\partial \eta_1} = 0 \text{ implies } \frac{\Gamma'(\hat{\eta}_1)}{\Gamma(\hat{\eta}_1)} + \ln \hat{\theta} = \frac{\sum_{i=n-m+1}^{n} \ln x_{(i)}}{m} \quad .$$

ii) for the case of known shape parameter η,

$$\frac{\partial K}{\partial \theta} = \frac{T}{\theta^2} - \frac{(n-m)\eta}{\theta} - \frac{mk^*\eta}{\theta} \text{ and}$$

$$\frac{\partial K}{\partial k^*} = \eta \sum_{i=n-m+1}^{n} \ln x_{(i)} - mn \frac{\Gamma'(k^*\eta)}{\Gamma(k^*\eta)} - m\eta \ln \theta$$

and setting these equal to zero, we obtain

$$\hat{\theta} = \frac{T}{\eta(n-m+mk^*)} \qquad \text{and}$$

$$\ell(\hat{k}^*) = \frac{\Gamma'(\hat{k}^*\eta)}{\Gamma(\hat{k}^*\eta)} + \ln \hat{\theta} - \frac{\sum_{i=n-m+1}^{n} \ln x_{(i)}}{m} = 0 \; .$$

iii) for the case of known θ, we may use $Y_i = \frac{X_i}{\theta}$ and hence obtain

$$\frac{\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} = \frac{\sum_{i=1}^{n-m} \ln y_{(i)}}{n-m}$$

and

$$\frac{\Gamma'(\hat{\eta}_1)}{\Gamma(\hat{\eta}_1)} = \frac{\sum_{i=n-m+1}^{n} \ln y_{(i)}}{m} \; .$$

iv) for the case of known k^*

$$\frac{\partial K}{\partial \theta} = \frac{T}{\theta^2} - \frac{(n-m)\eta}{\theta} - \frac{mk^*\eta}{\theta}$$

$$\frac{\partial K}{\partial \eta} = \sum_{i=1}^{n-m} \ln x_{(i)} + k^* \sum_{i=n-m+1}^{n} \ln x_{(i)} - \frac{(n-m)\Gamma'(\eta)}{\Gamma(\eta)} - \frac{mk^*\Gamma'(k^*\eta)}{\Gamma(k^*\eta)}$$

and thus $\hat{\theta} = \dfrac{T}{\eta\{n-m(1-k^*)\}}$ and

$$\sum_{i=1}^{n-m} \ln x_{(i)} + k^* \sum_{i=n-m+1}^{n} \ln x_{(i)} - \frac{(n-m)\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} - \frac{mk^*\Gamma'(k^*\hat{\eta})}{\Gamma(k^*\hat{\eta})} = 0.$$

CHAPTER IV

The Lognormal Distribution

We now examine the lognormal distribution as a life-testing distribution that is a competitor to the gamma distribution. We shall show that the exchangeable model involving the lognormal family of distributions indexed by the shape parameter σ is outlier-resistant completely on the right. We shall then determine that as heterogeneity increases the spurious observation tends to appear as an outlier and we shall determine where it will most likely appear. We shall also consider estimation in the presence of this outlier.

4.1 Characteristics of the Lognormal Distribution.

If a random variable X has the lognormal distribution with location parameter μ and shape parameter σ, then the probability density function of X is given by

$$f(x;\mu,\sigma) = \begin{cases} \dfrac{1}{\sigma x\sqrt{2\pi}} \exp\{-\dfrac{1}{2\sigma^2}(\ell n\ x-\mu)^2\}, & x > 0,\ \sigma > 0,\ -\infty < \mu < \infty \\ 0, & \text{elsewhere} \end{cases} .$$

We shall write $X \sim \Lambda(\mu,\sigma)$. The following table summarizes some some of its properties:

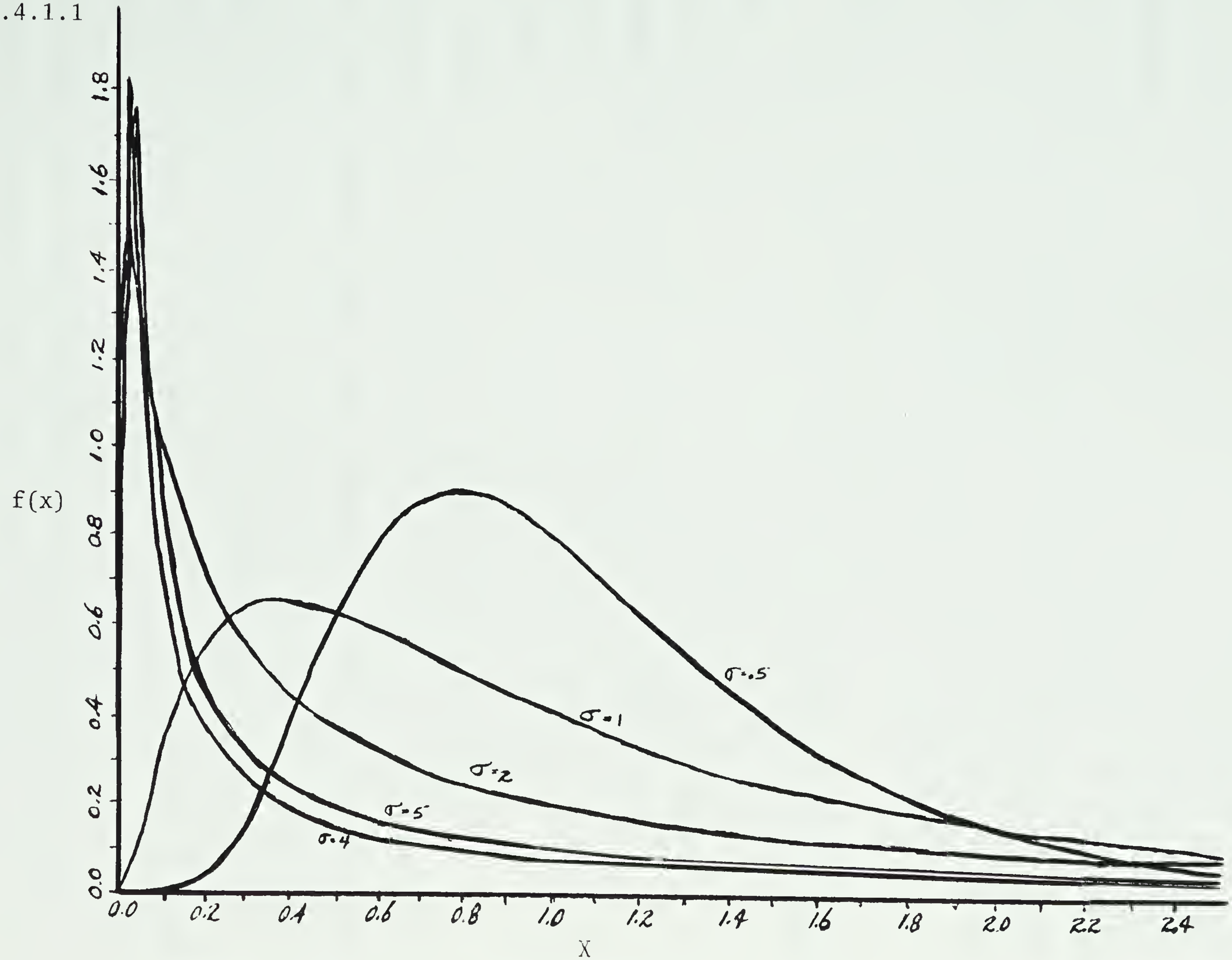
Table 4.1.1

mean	$e^{\mu+\frac{1}{2}\sigma^2}$
variance	$e^{\sigma^2}(e^{\sigma^2}-1)e^{2\mu}$
mode	$e^{\mu-\sigma^2}$
median	e^{μ}
coefficient of variation	$(e^{\sigma^2}-1)^{1/2}$
skewness	$(e^{\sigma^2}+2)(e^{\sigma^2}-1)^{1/2}$
kurtosis	$\omega^4+2\omega^3+3\omega^2-6,\ \omega = e^{\sigma^2}$

The lognormal distribution has monotone likelihood ratio (MLR) in $T_1(x) = \ell n\ x$, σ known and in $T_2(x) = (\ell n\ x-\mu)^2$, μ known but not

MLR in x for μ known and nonzero. $T_1 = \sum_{i=1}^{n} \ln x_i$ is a complete sufficient statistic for μ (σ known) and $(T_2 = \sum_{i=1}^{n} (\ln x_i)^2,\ T_1 = \sum_{i=1}^{n} \ln x_i)$ are jointly complete sufficient statistics for (μ, σ^2). $T_3 = \sum_{i=1}^{n} (\ln x_i - \mu)^2$ is sufficient for σ^2 (μ known). The lognormal distribution can assume shapes from severely right-skewed to essentially symmetrical.

Fig.4.1.1



Lognormal p.d.f. for various values of σ (μ =0)

If $X \sim \Lambda(\mu, \sigma)$ then $Y = \ln X \sim N(\mu, \sigma^2)$.

For some data, the lognormal distribution is a competitor to the gamma distribution. The lognormal p.d.f. is unimodal, vanishes at $x = 0$ and its mode is at $x = e^{-\sigma^2} < 1$. As σ^2 increases, the mode converges to zero. From the above graph where $\sigma = 2$, the lognormal p.d.f. appears monotonically decreasing for almost all $x > 0$ and when $\sigma = 5$ the mode is virtually zero. This may be compared to the gamma distribution with shape parameter $\eta < 1$. Here the density is infinite at zero and monotonically decreasing thereafter. The lognormal distribution has a non-monotonic failure rate.

Neyman and Scott (1971) document data from rain-making experiments where the data is nonzero rainfall per experimental unit (an experimental day or storm). Here the distributions are reverse J-shaped with long "tails" and frequently display substantial "outliers". They suggest that an outlier-prone distribution such as the gamma or lognormal might appropriately model this data.

Nelson (1977) points out the usefulness of the lognormal distribution for approximating distributions of input variables such as costs, sales, market shares, etc. required for Monte Carlo simulations of business decisions. This distribution has been used by Howard and Dodson (1961) and Peck (1961) to study semiconductor devices and by Goldwaithe (1961) in small-particle statistical economics and biology. Singpurwalla and Keubler (1966) used it to study the lifetimes of high speed steel drills since the failure rate first increases and then decreases, indicating the drill could resharpen itself and prolong its life. It has wide applicability to reliability, especially maintainablity and fracture problems. If $X_1 < X_2 < \ldots < X_n$ denote

sizes of a fatigue crack at successive stages of its growth and X_0 the initial size of the crack, assuming a "proportional effect model" for growth of the crack (Kao (1965)), this implies crack growth at stage i, $X_i - X_{i-1}$, is randomly proportional to the size of the crack, X_{i-1} (i.e. $X_i - X_{i-1} = \pi_i X_{i-1}$, $i = 1,2,\ldots,n$ where π_i are independent), and the item fails when crack size reaches X_n. It can be shown that $\ell n\, X_n$ is asymptotically normal and hence X_n is lognormal. This model is also true for the distribution of oil pool sizes by the same argument on how the pools are initially formed.

4.2 Outlier-proneness of the exchangeable model with the Lognormal distribution

Neyman and Scott (1971) have demonstrated that for i.i.d.r.v.'s the family of lognormal distributions indexed by σ is outlier-prone completely on the right. If we now consider the exchangeable model with at most one outlier, we have n-1 observations from $\Lambda(\mu,\sigma)$ and one observation from $\Lambda(\mu_1,\sigma)$, $\mu_1 \geq \mu$ (Case I) or n-1 observations from $\Lambda(\mu,\sigma)$ and one observation from $\Lambda(\mu,\sigma_1)$, $\sigma_1 \geq 0$ (Case II).

Case I: Scale change

We first consider the exchangeable model based on the lognormal distribution with possible change in μ. Then $f(x;\underline{\theta})$ is $\Lambda(\mu,\sigma)$ and $f(x;\underline{\xi})$ is $\Lambda(\mu_1,\sigma)$ where $\mu_1 = k^*\mu$, $k^* \geq 1$. Kale (1975b) proved that the exchangeable model with (at most) one possible outlier observation for scale parameter families for non-negative random variables involving a possible change in scale is outlier-prone completely on the right. Thus the model we are considering would be outlier-prone.

Case II: Shape Change

Consider the exchangeable model based on the lognormal distribution with possible change in shape parameter σ. Then $f(x;\underline{\theta})$ is $\Lambda(0,\sigma)$ and $f(x;\underline{\xi})$ is $\Lambda(0,\sigma_1)$ where $\sigma_1^2 = k^*\sigma^2$, $k^* \geq 1$. The likelihood may be written as

$$L(\underline{x};\sigma,k^*) = \frac{1}{n}\sum_{r=1}^{n}\prod_{i\neq r} f(x_i;\underline{\theta})f(x_r;\underline{\xi}),\ x_i > 0\ ,\ k^* \geq 1,$$

$$\sigma_1^2 = k^*\sigma^2\ ,\ i = 1,2,\ldots,n$$

($k^* \geq 1$ since we are considering $x_{(n)}$ as a possible outlier). Then the joint density of the order statistics may be written as

$$f(x_{(1)},\ldots,x_{(n)}) = \frac{1}{n}\, n! \sum_{r=1}^{n} \frac{f(x_r;\sigma_1)}{f(x_r;\sigma)} \prod_{i=1}^{n} f(x_i,\sigma)$$

and

$$g(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= \int_{x_{(1)}}^{x_{(n-1)}} \int_{x_{(1)}}^{x_{(n-2)}} \cdots \int_{x_{(1)}}^{x_{(3)}} f(x_{(1)},\ldots,x_{(n)})dx_{(2)}\cdots dx_{(n-2)}$$

$$= (n-1)! \sum_{r=1}^{n} h_r(x_{(1)},x_{(n-1)},x_{(n)})$$

where $h_r(x_{(1)},x_{(n-1)},x_{(n)}) = \int_{S_{n-3}} \frac{f(x_r;\sigma_1)}{f(x_r;\sigma)} \prod_{i=1}^{n} f(x_i;\sigma)dx_2 \ldots dx_{n-2}$

and S_{n-3} is the region $x_{(1)} < x_{(2)} < \cdots < x_{(n-2)} < x_{(n-1)}$.

For $r = 2,\ldots,n-2$

$$h_r(x_{(1)},x_{(n-1)},x_{(n)}) = \frac{f(x_{(1)};\sigma)f(x_{(n-1)};\sigma)f(x_{(n)};\sigma)}{(r-2)!(n-r-2)!}$$

(continued)

$$\cdot \int_{x_{(1)}}^{x_{(n-1)}} [F(x_{(r)};\sigma)-F(x_{(1)};\sigma)]^{r-2}[F(x_{(n-1)};\sigma)-F(x_{(r)};\sigma)]^{n-r-2}$$

$$f(x_{(r)};\sigma_1)dx_{(r)} \ ,$$

while

$$h_1(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= f(x_{(1)};\sigma_1)f(x_{(n-1)};\sigma)f(x_{(n)};\sigma) \ \frac{[F(x_{(n-1)};\sigma)-F(x_{(1)};\sigma)]^{n-3}}{(n-3)!} \ ,$$

$$h_{n-1}(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= f(x_{(1)};\sigma)f(x_{(n-1)};\sigma_1) \ f(x_{(n)};\sigma) \ \frac{[F(x_{(n-1)};\sigma)-F(x_{(1)};\sigma)]^{n-3}}{(n-3)!} \ ,$$

and

$$h_n(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= f(x_{(1)};\sigma)f(x_{(n-1)};\sigma)f(x_{(n)};\sigma_1) \ \frac{[F(x_{(n-1)};\sigma)-F(x_{(1)};\sigma)]^{n-3}}{(n-3)!} \ .$$

Thus, letting $t(y) = f(y;\sigma)$ and $s(y) = f(y;\sigma_1)$

$$(4.2.1)\ .\ .\ .\ .\quad g(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= (n-1)!\{t(x_{(1)})t(x_{(n-1)})t(x_{(n)})$$

$$\times \sum_{r=2}^{n-2} \int_{x_{(1)}}^{x_{(n-1)}} \frac{[T(x_{(r)})-T(x_{(1)})]^{r-2}[T(x_{(n-1)})-T(x_{(r)})]^{n-r-2}s(x_{(r)})}{(r-2)!(n-r-2)!}\, dx_{(r)}$$

$$+[s(x_{(1)})t(x_{(n-1)})t(x_{(n)})+t(x_{(1)})s(x_{(n-1)})t(x_{(n)})$$

$$+t(x_{(1)})t(x_{(n-1)})s(x_{(n)})]\ \frac{[T(x_{(n-1)})-T(x_{(1)})]^{n-3}}{(n-3)!}\} .$$

In 4.2.1, we may replace the letter of integration in the definite integrals by z and use the fact that

$$(n-1)! \sum_{r=2}^{n-2} \frac{\alpha^{r-2}\beta^{n-r-2}}{(r-2)!(n-r-2)!} = \frac{(n-1)!}{(n-4)!}\ (\alpha+\beta)^{n-4}$$

to give

$$g(x_{(1)},x_{(n-1)},x_{(n)}) = \frac{(n-1)!}{(n-3)!}\ \{(n-3)t(x_{(1)})t(x_{(n-1)})t(x_{(n)}) \times$$

$$[T(x_{(n-1)})-T(x_{(n)})]^{n-4}[S(x_{(n-1)})-S(x_{(1)})]$$

$$+\ [s(x_{(1)})t(x_{(n-1)})t(x_{(n)})+t(x_{(1)})s(x_{(n-1)})t(x_{(n)})$$

(continued)

$$+ t(x_{(1)})t(x_{(n-1)})s(x_{(n)})][T(x_{(n-1)})-T(x_{(1)})]^{n-3}\}$$

$$= \frac{(n-1)!}{(n-3)!}\{T(x_{(n-1)})-T(x_{(1)})\}^{n-4}\{(n-3)t(x_{(1)})t(x_{(n-1)})t(x_{(n)})\times$$

$$[S(x_{(n-1)})-S(x_{(1)})] + [s(x_{(1)})t(x_{(n-1)})t(x_{(n)})+t(x_{(1)})s(x_{(n-1)})t(x_{(n)})$$

$$+ t(x_{(1)})t(x_{(n-1)})s(x_{(n)})][T(x_{(n-1)})-T(x_{(1)})]\}.$$

Let

$$p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

$$= \int_0^\infty \int_0^{x_{(n-1)}} \int_{(k+1)x_{(n-1)}-kx_{(1)}}^\infty g(x_{(1)},x_{(n-1)},x_{(n)})dx_{(n)}dx_{(1)}dx_{(n-1)}$$

and let $x = x_{(1)}$ and $y = x_{(n-1)}$.

Theorem 4.2.1: If $p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$ then, for non-negative random variables,

$$p \leq 1-k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

where S and s are, respectively, the distribution function and probability density function of the spurious observation and T and t

are, respectively, the distribution function and probability density function of the non-spurious observations.

Proof:

$$p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

$$= \int_0^\infty \int_0^{x_{(n-1)}} \int_{(k+1)x_{(n-1)}-kx_{(1)}}^\infty g(x_{(1)},x_{(n-1)},x_{(n)})dx_{(n)}dx_{(1)}dx_{(n-1)}$$

$$= \frac{(n-1)!}{(n-3)!} \int_0^\infty \int_0^{x_{(n-1)}} \{T(x_{(n-1)})-T(x_{(1)})\}^{n-4}$$

$$\Big\{(n-3)t(x_{(1)})t(x_{(n-1)})[1-T\{(k+1)x_{(n-1)}-kx_{(1)}\}][S(x_{(n-1)})-S(x_{(1)})]$$

$$+ \Big(s(x_{(1)})t(x_{(n-1)})[1-T\{(k+1)x_{(n-1)}-kx_{(1)}\}]$$

$$+ t(x_{(1)})s(x_{(n-1)})[1-T\{(k+1)x_{(n-1)}-kx_{(1)}\}]$$

$$+ t(x_{(1)})t(x_{(n-1)})[1-S\{(k+1)x_{(n-1)}-kx_{(1)}\}]\Big)$$

$$\times [T(x_{(n-1)})-T(x_{(1)})]\Big\}dx_{(1)}dx_{(n-1)}$$

$$= \frac{(n-1)!}{(n-3)!} \{I_1+I_2+I_3+I_4 - (I_5+I_6+I_7+I_8)\}.$$

Let $x = x_{(1)}$, $y = x_{(n-1)}$

$$I_1 = \int_0^\infty \int_0^y (n-3)t(x)t(y)\{T(y)-T(x)\}^{n-4}\{S(y)-S(x)\}dxdy$$

$$= (n-3)\int_0^\infty S(y)t(y)\int_0^y t(x)\{T(y)-T(x)\}^{n-4}dxdy$$

$$-(n-3)\int_0^\infty t(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-4}dydx$$

$$= (n-3)\int_0^\infty S(y)t(y)\frac{\{T(y)\}^{n-3}}{(n-3)}\,dy - (n-3)\int_0^\infty S(x)t(x)\frac{\{1-T(x)\}^{n-3}}{(n-3)}\,dx$$

$$= \int_0^\infty t(x)S(x)[\{T(x)\}^{n-3}-\{1-T(x)\}^{n-3}]dx \quad .$$

$$I_2 = \int_0^\infty \int_0^y \{T(y)-T(x)\}^{n-3}s(x)t(y)dxdy$$

$$= \int_0^\infty s(x)\int_x^\infty \{T(y)-T(x)\}^{n-3}t(y)dydx$$

$$= \int_0^\infty \frac{s(x)}{(n-2)}\{1-T(x)\}^{n-2}dx.$$

Integrating by parts where $u = \{1-T(x)\}^{n-2}$ $\quad dv = s(x)dx$

$$du = -(n-2)t(x)\{1-T(x)\}^{n-3}dx \qquad v = S(x)$$

then

$$I_2 = \int_0^\infty S(x)t(x)\{1-T(x)\}^{n-3}dx.$$

$$I_3 = \int_0^\infty \int_0^y \{T(y)-T(x)\}^{n-3}t(x)s(y)dxdy$$

$$= \int_0^\infty s(y) \int_0^y \{T(y)-T(x)\}^{n-3}t(x)dxdy$$

$$= \int_0^\infty s(x) \frac{\{T(x)\}^{n-2}}{(n-2)} dx \quad .$$

Integrating by parts where $u = \{T(x)\}^{n-2}$ $\quad dv = s(x)dx$

$du = (n-2)\{T(x)\}^{n-3}t(x)dx \quad v = S(x)$

$$I_3 = \frac{1}{n-2} - \int_0^\infty S(x)t(x)\{T(x)\}^{n-3}dx \quad .$$

$$I_4 = \int_0^\infty \int_0^y t(x)t(y)\{T(y)-T(x)\}^{n-3}dx$$

$$= \int_0^\infty t(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-3}dydx$$

$$= \int_0^\infty t(x) \frac{\{1-T(x)\}^{n-2}}{(n-2)} dx$$

$$= \frac{1}{(n-1)(n-2)} \quad .$$

Therefore $I_1 + I_2 + I_3 + I_4 = \frac{1}{(n-2)} + \frac{1}{(n-1)(n-2)}$

$$= \frac{n}{(n-1)(n-2)} \quad .$$

$$I_5 = \int_0^\infty \int_0^y (n-3)t(x)t(y)\{T(y)-T(x)\}^{n-4}\ T\{(k+1)y-k(x)\}\{S(y)-S(x)\}dxdy$$

$$\geq \int_0^\infty \int_0^y (n-3)t(x)t(y)\{T(y)-T(x)\}^{n-4}T(y)\{S(y)-S(x)\}dxdy$$

$$= \int_0^\infty (n-3)t(y)T(y)S(y)\int_0^y t(x)\{T(y)-T(x)\}^{n-4}dxdy$$

$$- \int_0^\infty (n-3)t(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-3}dydx$$

$$- \int_0^\infty (n-3)t(x)T(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-4}dydx$$

$$= \int_0^\infty t(x)T(x)S(x)\{T(x)\}^{n-3}dx - \int_0^\infty \frac{(n-3)}{(n-2)}\ t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$- \int_0^\infty t(x)T(x)S(x)\{1-T(x)\}^{n-3}dx.$$

Therefore

$$I_5 \geq \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx - \frac{(n-3)}{(n-2)} \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$- \int_0^\infty t(x)T(x)S(x)\{1-T(x)\}^{n-3}dx \quad .$$

$$I_7 \geq \int_0^\infty \int_0^y t(x)s(y)T(y)\{T(y)-T(x)\}^{n-3}dxdy$$

$$= \int_0^\infty s(y)T(y) \int_0^y t(x)\{T(y)-T(x)\}^{n-3}dxdy$$

$$= \int_0^\infty s(x)T(x) \frac{\{T(x)\}^{n-2}}{(n-2)} dx$$

$$= \int_0^\infty s(x) \frac{\{T(x)\}^{n-1}}{(n-2)} dx \quad .$$

Integrating by parts where $u = \{T(x)\}^{n-1}$ $\quad dv = s(x)dx$

$du = (n-1)t(x)\{T(x)\}^{n-2}dx \quad v = S(x)$

$$I_7 \geq \frac{1}{n-2} - \frac{(n-1)}{(n-2)} \int_0^\infty t(x)S(x)\{T(x)\}^{n-2} dx \quad .$$

$$I_8 \geq \int_0^\infty \int_0^y t(x)t(y)S(y)\{T(y)-T(x)\}^{n-3}dxdy$$

$$= \int_0^\infty \frac{t(y)S(y)}{(n-2)} \{T(y)\}^{n-2}dy$$

$$= \frac{1}{(n-2)} \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx.$$

$$I_6 = \int_0^\infty t(y)\int_0^y s(x)\{T(y)-T(x)\}^{n-3}T\{(k+1)y-kx\}dxdy \quad .$$

Integrating by parts where

$$u = \{T(y)-T(x)\}^{n-3}T\{(k+1)y-kx\} \qquad dv = s(x)dx$$

$$du = \{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}\{-kdx\} \qquad v = S(x)$$

$$+ \quad (n-3)\{T(y)-T(x)\}^{n-4}T\{(k+1)y-kx\}\{-t(x)\}dx$$

$$I_6 = k \int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$+ (n-3)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-4}T\{(k+1)y-kx\}t(x)dxdy$$

$$\geq k \int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$+ (n-3)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-4}T(y)t(x)dxdy \quad .$$

But

$$(n-3)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-4}T(y)t(x)dxdy$$

$$= (n-3)\int_0^\infty t(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-3}dydx$$

$$+ (n-3)\int_0^\infty t(x)T(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-4}dydx$$

$$= \frac{(n-3)}{(n-2)}\int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$+ \int_0^\infty t(x)T(x)S(x)\{1-T(x)\}^{n-3}dx$$

$$= \frac{(n-3)}{(n-2)}\int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx - \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$+ \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-3}dx.$$

Therefore $I_5 + I_6 + I_7 + I_8 \geq \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx$

$$- \frac{(n-3)}{(n-2)}\int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx + \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

(continued)

$$- \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-3}dx$$

$$+ k\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$+ \frac{(n-3)}{(n-2)} \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx - \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$+ \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-3}dx + \frac{1}{(n-2)}$$

$$- \frac{(n-1)}{(n-2)} \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx \quad + \frac{1}{(n-2)} \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx$$

Therefore

$$p \leq (n-1)(n-2)\Big[\frac{n}{(n-1)(n-2)} - \frac{1}{(n-2)}$$

$$- k\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy\Big]$$

$$= 1 - k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)[T(y)-T(x)]^{n-3}t\{(k+1)y-kx\}dxdy \ .$$

Theorem 4.2.2: The exchangeable model with (at most) one spurious observation based on the lognormal family of distributions indexed by shape parameter σ is outlier-resistant completely on the right.

Proof: From Theorem 4.2.1 we know that if

$$p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

then, for non-negative random variables,

$$p \leq 1 - k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

where $x = x_{(1)}$, $y = x_{(n-1)}$, S and s are, respectively, the distribution function and p.d.f. of a spurious observation, and T and t are, respectively, the distribution function and p.d.f. of a non-spurious observation. If, for all $k > 0$, $n > 2$ sup $p < 1$ then the family is outlier-resistant completely on the right.

It is then sufficient to show that

$$(4.2.2)\quad H = k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$> 0.$$

Let $s_1 = \dfrac{\ln x}{\sigma}$ $\qquad s_2 = \dfrac{\ln y}{\sigma}$ $\qquad v = \dfrac{\sigma}{\sigma_1}$

$ds_1 = \dfrac{dx}{x\sigma}$ $\qquad ds_2 = \dfrac{dy}{\sigma y}$

If ϕ and Φ respectively denote the standardized normal p.d.f. and d.f. then expression (4.2.2) becomes

$$k(n-1)(n-2)\int_{-\infty}^{\infty}\phi(s_2)\int_{-\infty}^{s_2}\Phi(vs_1)\{\Phi(s_2)-\Phi(s_1)\}^{n-3}\phi\{\frac{\ln[(k+1)e^{\sigma s_2}-ke^{\sigma s_1}]}{\sigma}\}$$

$$\times\frac{e^{\sigma s_1}ds_1ds_2}{(k+1)e^{\sigma s_2}-ke^{\sigma s_1}}$$

$$\geq k(n-1)(n-2)\int_3^5\phi(s_2)\int_0^2\frac{\Phi(vs_1)e^{\sigma s_1}}{(k+1)e^{\sigma s_2}-ke^{\sigma s_1}}\{\Phi(s_2)-\Phi(s_1)\}^{n-3}$$

$$\times\ \phi\{\frac{\ln[(k+1)e^{\sigma s_2}-ke^{\sigma s_1}]}{\sigma}\}ds_1ds_2$$

In this region, $s_2 \geq 3$, $s_1 \geq 0$, $s_2 \geq 1 + s_1$. Therefore $\Phi(vs_1) \geq 1/2$ for all v(i.e. for all σ_1), $e^{\sigma s_1} \geq 1$,

$$\Phi(s_2) - \Phi(s_1) \geq \Phi(3) - \Phi(2) \text{ and } e^{-\sigma s_2} \geq e^{-5\sigma}.$$

Then

$$H \geq \frac{k(n-1)(n-2)}{2(k+1)}\{\Phi(3) - \Phi(2)\}^{n-3}\frac{1}{2}e^{-5\sigma}\,2\int_3^5\phi(s_2)\phi\left(s_2+\frac{\ln(k+1)}{\sigma}\right)ds_2\ .$$

Let $u = \dfrac{\ln(k+1)}{\sigma}$. Then $u > 0$ since $k > 0$ and

$$\int_3^5\phi(s_2)\phi(s_2+u)ds_2 = \frac{1}{\sqrt{2\pi}}\int_3^5\frac{1}{\sqrt{2\pi}}e^{-1/2\left[\frac{s_2+u/2}{1/\sqrt{2}}\right]^2-u^2/4}ds_2$$

$$= \frac{e^{-u^2/4}}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_{\sqrt{2}(3+u/2)}^{\sqrt{2}(5+u/2)} \phi(z)dz$$

$$= \frac{e^{-u^2/4}}{2\sqrt{\pi}} \left[\Phi\{\sqrt{2}(5+u/2)\} - \Phi\{\sqrt{2}(3+u/2)\}\right]$$

$$> 0.$$

Then for all $k > 0$, $n > 2$, $v = \sigma/\sigma_1 > 0$, $H > 0$ and $p = 1-H < 1$ and therefore $\sup p < 1$. Thus the exchangeable model with (at most) one spurious observation involving the lognormal family of distributions indexed by shape parameter σ is outlier-resistant completely on the right.

4.3 Detection of Outliers

Case I: Scale change.

If $n-1$ observations are from $\Lambda(\mu,\sigma)$ and one observation is from $\Lambda(\mu_1,\sigma)$, $\mu_1 = k^*\mu$, $k^* \geq 1$ then

$$\Psi(x) = \frac{dG(x)}{dF(x)} = \frac{\frac{1}{\sigma x\sqrt{2\pi}}\, e^{-1/2\left(\frac{\ln x-\mu_1}{\sigma}\right)^2}}{\frac{1}{\sigma x\sqrt{2\pi}}\, e^{-1/2\left(\frac{\ln x-\mu}{\sigma}\right)^2}}$$

$$= \exp\left[-\frac{1}{2\sigma^2}\{(\ln x-\mu_1)^2-(\ln x-\mu)^2\}\right]$$

$$= \left[\exp\left\{-\frac{(\mu_1^2-\mu^2)}{2\sigma^2}\right\}\right] x^{\frac{\mu_1-\mu}{\sigma^2}}$$

and $$\Psi'(x) = \left(\frac{\mu_1-\mu}{\sigma^2}\right) x^{\frac{\mu_1-\mu}{\sigma^2}-1} \exp\left\{-\frac{1}{2\sigma^2}(\mu_1^2-\mu^2)\right\} .$$

Thus $$\Psi'(x) \begin{cases} > 0 & \text{if } \mu_1 > \mu \\ < 0 & \text{if } \mu_1 < \mu . \end{cases}$$

Consequently $X_{(n)}$ has maximum probability of being the spurious observation if $\mu_1 > \mu$; $X_{(1)}$ has maximum probability of being the

spurious observation if $\mu_1 < \mu$.

Case II: Shape change

If $n-1$ observations are from $\Lambda(\mu,\sigma)$ and one observation is from $\Lambda(\mu,\sigma_1)$, $\sigma_1^2 = k^*\sigma^2$, $k^* \geq 1$ then

$$\Psi(x) = \frac{dG(x)}{dF(x)} = \frac{\sigma}{\sigma_1} \exp\left\{-\frac{1}{2}(\ln x-\mu)^2\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma^2}\right)\right\}$$

is not monotone in x.

Consider $u(r;n,\sigma,k^*) = P(X_{(r)}$ is the spurious observation) (i.e. $X_{(r)}$ is the order statistic whose distribution has parameter σ_1).

Then

$$u(r;n,\sigma,k^*) = \binom{n-1}{r-1}\int_0^\infty \{F(y;\sigma)\}^{r-1}\{1-F(y;\sigma)\}^{n-r} f(y;\sigma_1)dy$$

$$= \binom{n-1}{r-1}\int_0^\infty \left\{\Phi\left(\frac{\ln y-\mu}{\sigma}\right)\right\}^{r-1}\left\{1-\Phi\left(\frac{\ln y-\mu}{\sigma}\right)\right\}^{n-r} \frac{\exp -\frac{1}{2}\left(\frac{\ln y-\mu}{\sigma_1}\right)^2}{\sigma_1 y\sqrt{2\pi}}dy$$

where Φ is the standard normal distribution function

$$= \binom{n-1}{r-1} \int_0^{\infty} \left[\Phi\left\{\left(\frac{\ln y-\mu}{\sigma_1}\right)\sqrt{k^*}\right\}\right]^{r-1} \left[1-\Phi\left\{\left(\frac{\ln y-\mu}{\sigma}\right)\sqrt{k^*}\right\}\right]^{n-r}$$

$$\frac{\exp - \frac{1}{2}\left(\frac{\ln y-\mu}{\sigma_1}\right)^2}{\sigma_1 y\sqrt{2\pi}} dy$$

and setting $x = \dfrac{\ln y-\mu}{\sigma_1}$ we have

$$u(r;n,\sigma,k^*) = \binom{n-1}{r-1} \int_{-\infty}^{\infty} \{\Phi(x\sqrt{k^*})\}^{r-1} \{1-\Phi(x\sqrt{k^*})\}^{n-r} \frac{\exp\{-\frac{x^2}{2}\}}{\sqrt{2\pi}} dx .$$

Now

$$0 \leq \Phi(x\sqrt{k^*}) \leq 1 \quad \text{and} \quad \lim_{k^*\to\infty} \Phi(x\sqrt{k^*}) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0 \end{cases} .$$

Therefore

$$\lim_{k^*\to\infty} u(r;n,\sigma,k^*) = \binom{n-1}{r-1} \lim_{k^*\to\infty} \Big\{ \int_0^{\infty} \{\Phi(x\sqrt{k^*})\}^{r-1} \{1-\Phi(x\sqrt{k^*})\}^{n-r} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

$$+ \int_{-\infty}^{0} \{\Phi(x\sqrt{k^*})\}^{r-1} \{1-\Phi(x\sqrt{k^*})\}^{n-r} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \Big\}.$$

By the dominated convergence theorem

$$\lim_{k^*\to\infty} u(n;n,\sigma,k^*) = \int_0^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = 1/2$$

$$\lim_{k^*\to\infty} u(1;n,\sigma,k^*) = \int_{-\infty}^0 \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = 1/2$$

and for any other $r = 2,\ldots,n-1$

$$\{\Phi(x\sqrt{k^*})\}^{r-1}\{1-\Phi(x\sqrt{k^*})\}^{n-r} \to 0 \quad \text{as} \quad k^* \to \infty ,$$

so that, by the dominated convergence theorem,

$$\lim_{k^*\to\infty} u(r;n,\sigma,k^*) = \binom{n-1}{r-1}\int_{-\infty}^{\infty} 0\, \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \;\; = 0,\ r = 2,\ldots,n-1.$$

It thus appears that the spurious observation tends to occur at either $X_{(1)}$ or $X_{(n)}$ with equal probability (approaching 1/2) as the shape parameter of the spurious observation increases.

4.4 Estimation for Lognormal parameters

4.4.1 Standard Estimators

If $X \sim \Lambda(\mu,\sigma)$, M.L.E.'s of μ and σ may be obtained from M.L.E. of $W = \ln X$ since $W \sim N(\mu,\sigma^2)$. Thus the M.L.E.'s of μ and σ are

$$\hat{\mu} = \frac{\sum_{i=1}^{n} W_{(i)}}{n} = \frac{\sum_{i=1}^{n} \ln X_{(i)}}{n}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} \left(\ln X_{(i)}\right)^2 - n\left(\frac{\sum_{i=1}^{n} \ln X_{(i)}}{n}\right)^2}{n}$$

Since $\hat{\mu}$ and $\hat{\sigma}^2$ are based on complete sufficient statistics, we can obtain best linear estimators $\hat{\mu}$ and $s^2 = \frac{n\hat{\sigma}^2}{n-1}$ and $\frac{n-1}{n+1}\, s^2$ has minimum MSE among estimators of σ^2 of the form ks^2. The BLIE estimators of μ and σ^2 are $\hat{\mu}$ and $\tilde{\sigma}^2 = \frac{n}{n+1}\,\hat{\sigma}^2$. Asymptotically, all three estimators are equivalent. However, using the property of MLE's that if $\hat{\gamma}$ is MLE of γ, $g(\hat{\gamma})$ is MLE of $g(\gamma)$, we may obtain M.L.E.'s of the mean and variance of the lognormal to be respectively, $e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}$ and $e^{2\hat{\mu}+\hat{\sigma}^2}(e^{\hat{\sigma}^2}-1)$.

To obtain confidence bounds for parameters of the two-parameter lognormal distributions, we use

$$\hat{\mu} = \bar{W} \sim N(\mu, \sigma^2/n).$$

Thus, for σ^2 known, a $(1-\alpha)$ 100% confidence interval (C.I.) for μ is given by

$$\bar{w} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{w} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} .$$

For σ^2 unknown, $\frac{n\hat{\sigma}^2}{\sigma^2} = \frac{\sum_{i=1}^{n} w_i^2 - n\bar{w}^2}{\sigma^2} \sim \chi^2_{n-1}$ df

and $\left(\frac{\bar{w}-\mu}{\sigma/\sqrt{n}}\right) \Big/ \sqrt{\frac{n\sigma^2}{\sigma^2(n-1)}} \sim t_{n-1}$df

and a two-sided $(1-\alpha)$ 100% C.I. for μ is

$$\bar{w} + t_{\alpha/2,n-1\mathrm{df}} \frac{\hat{\sigma}}{\sqrt{n-1}} \leq \mu \leq \bar{w} + t_{1-\alpha/2,n-1\mathrm{df}} \frac{\hat{\sigma}}{\sqrt{n-1}} .$$

4.4.2 Estimators suggested for use

Case I: Scale change

Consider the estimator $\hat{\mu} = \frac{\sum_{i=1}^{n} \ln X_i}{n}$. Under the homogeneous model, $E(\hat{\mu}) = \mu$ and $\mathrm{Var}(\hat{\mu}) = \frac{\sigma^2}{n}$. Thus its $\mathrm{MSE}(\hat{\mu}) = \frac{\sigma^2}{n}$. Under

the exchangeable model with (at most) one possible outlier,

$$E_{het}(\hat{\mu}) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{n-1}{n} \mu + \frac{1}{n} \mu_1\right)$$

$$= \mu\{1 + \frac{k^*-1}{n}\}$$

and

$$Var_{het}(\hat{\mu}) = \frac{1}{n^2} \sum_{i=1}^{n} Var_{het} (\ell n \; X_i)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \left(\frac{n-1}{n} \sigma^2 + \frac{1}{n} \sigma^2\right)$$

$$= \frac{\sigma^2}{n} \; .$$

We then obtain $\quad MSE_{het}(\hat{\mu}) = Var_{het}(\hat{\mu}) + \{Bias_{het}(\hat{\mu})\}^2$

$$= \frac{\sigma^2}{n} + \left\{\frac{\mu(k^*-1)}{n}\right\}^2$$

and, as $k^* \to \infty$, $MSE_{het}(\hat{\mu}) \to \infty$. Thus $\hat{\mu} = \dfrac{\sum_{i=1}^{n} \ell n \; X_i}{n}$ appears to be a poor estimator in this instance.

Assuming the exchangeable model with m outliers (generally $m \leq 10\%$ of the sample size), we have $n-m$ observations with p.d.f. $f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp - \frac{1}{2\sigma^2}(\ln x-\mu)^2$ and m observations with p.d.f. $f(x;\mu_1,\sigma)$, $\mu_1 \geq \mu$. The joint likelihood may be written as

$$L(\underline{x};\mu,\mu_1,\sigma,I) = \frac{1}{\binom{n}{m}} \left\{ \frac{\exp - \frac{1}{2\sigma^2} \sum\limits_{x_i \notin I} (\ln x_i - \mu)^2}{\prod\limits_{x_i \notin I} \sigma x_i \sqrt{2\pi}} \right\} \left\{ \frac{\exp - \frac{1}{2\sigma^2} \sum\limits_{x_i \in I} (\ln x_i - \mu_1)^2}{\prod\limits_{x_i \in I} \sigma x_i \sqrt{2\pi}} \right\}$$

We wish to obtain $\max\limits_{\substack{I\in\mathcal{I} \\ \mu_1>\mu \\ \sigma>0}} L(\underline{x};\mu,\mu_1,\sigma,I)$ where $\mathcal{I}$ is the collection of all possible subsets of m outliers in a sample of size n. Kale (1974b) has shown for $\Psi(x) = \frac{dG(x)}{dF(x)}$ monotone increasing in x, $\hat{I} = \{x_{(n-m+1)},\ldots,x_{(n)}\}$ has maximum probability of being the set of spurious observations. For $\mu_1 > \mu$, $\Psi(x)$ is monotone increasing and hence

$$\max_{\substack{I\in\mathcal{I} \\ \mu_1>\mu \\ \sigma>0}} L(\underline{x};\mu,\mu_1,\sigma,I) = \max_{\substack{\mu_1>\mu \\ \sigma>0}} L(\underline{x};\mu,\mu_1,\sigma,\hat{I}) .$$

Now $K(\underline{x};\mu,\mu_1,\sigma,\hat{I}) = \ln L(\underline{x};\mu,\mu_1,\sigma,\hat{I})$

$$= C - \frac{1}{2\sigma^2}\left\{ \sum_{i=1}^{n-m} (\ln x_{(i)}-\mu)^2 + \sum_{i=n-m+1}^{n} (\ln x_{(i)}-\mu_1)^2 \right\}$$

$$- n/2 \ \ln 2\pi\sigma^2 - \sum_{i=1}^{n} \ln x_i \quad .$$

Then $$\frac{\partial K}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n-m} (\ln x_{(i)} - \mu)$$

$$\frac{\partial K}{\partial \mu_1} = \frac{1}{\sigma^2} \sum_{i=n-m+1}^{n} (\ln x_{(i)} - \mu_1)$$

$$\frac{\partial K}{\partial \sigma^2} = \frac{1}{\sigma^4} \left\{ \sum_{i=1}^{n-m} (\ln x_{(i)} - \mu)^2 + \sum_{i=n-m+1}^{n} (\ln x_{(i)} - \mu)^2 \right\} - \frac{n}{\sigma^2}$$

and hence $$\hat{\mu}_{het} = \sum_{i=1}^{n-m} \frac{\ln x_{(i)}}{n-m}$$

$$\hat{\mu}_{1_{het}} = \frac{\sum_{i=n-m+1}^{n} \ln x_{(i)}}{m}$$

$$\hat{\sigma}^2_{het} = \frac{\sum_{i=1}^{n-m} (\ln x_{(i)} - \hat{\mu}_{het})^2 + \sum_{i=n-m+1}^{n} (\ln x_{(i)} - \hat{\mu}_{1_{het}})^2}{n}$$

Thus it appears that maximum likelihood estimation suggests trimmed means as estimators of μ_1 and μ $(\mu_1 > \mu)$ and a pooled estimator, for σ^2, that utilizes these trimmed means.

Case II. Shape change

Under the homogeneous model, $E(\hat{\mu}) = E\left(\frac{\sum_{i=1}^{n} \ell n\, x_i}{n}\right) = \mu$ and $\mathrm{Var}(\hat{\mu}) = \frac{\sigma^2}{n}$. Under the exchangeable model with at most one possible outlier,

$$E_{het}(\hat{\mu}) = E\left(\frac{\sum_{i=1}^{n} \ell n\, x_i}{n}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{n-1}{n}\mu + \frac{1}{n}\mu\right) = \mu$$

and $$\mathrm{Var}_{het}(\hat{\mu}) = \mathrm{Var}\left(\frac{\sum_{i=1}^{n} \ell n\, x_i}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}_{het}(\ell n\, x_i)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(\frac{n-1}{n}\sigma^2 + \frac{1}{n}\sigma_1^2\right)$$

$$= \frac{\sigma^2(n-1+k^*)}{n^2} \quad .$$

Thus $$\mathrm{MSE}_{het}(\hat{\mu}) = \mathrm{Var}_{het}(\hat{\mu}) + \{\mathrm{Bias}_{het}(\hat{\mu})\}^2$$

$$= \frac{\sigma^2(n-1+k^*)}{n^2}$$

which tends to infinity as $k^* \to \infty$. Thus $\hat{\mu}$ appears to be a poor estimator of μ.

Since in this instance the spurious observation tends to appear at $X_{(1)}$ or at $X_{(n)}$, an estimator of μ based on the A-rule, W-rule, or S-rule (see Appendix I) would appear preferable to $\hat{\mu} = \frac{\sum_{i=1}^{n} \ln x_i}{n}$. Similar estimators for σ^2 appear preferable to $s^2_{\ln x}$.

CHAPTER V

The Weibull Distribution

We shall show that for i.i.d.r.v.'s the family of Weibull distributions, indexed by the shape parameter η, is outlier-prone completely on the right. On the other hand, we shall show that the exchangeable model based on the Weibull family of distributions indexed by the shape parameter is outlier-resistant completely on the right. For this exchangeable model, we shall determine which observation is most likely to be the spurious one and find asymptotic limits for these probabilities as heterogeneity increases. Also we shall examine the problem of estimation in the presence of outliers.

5.1 Characteristics of the Weibull Distribution

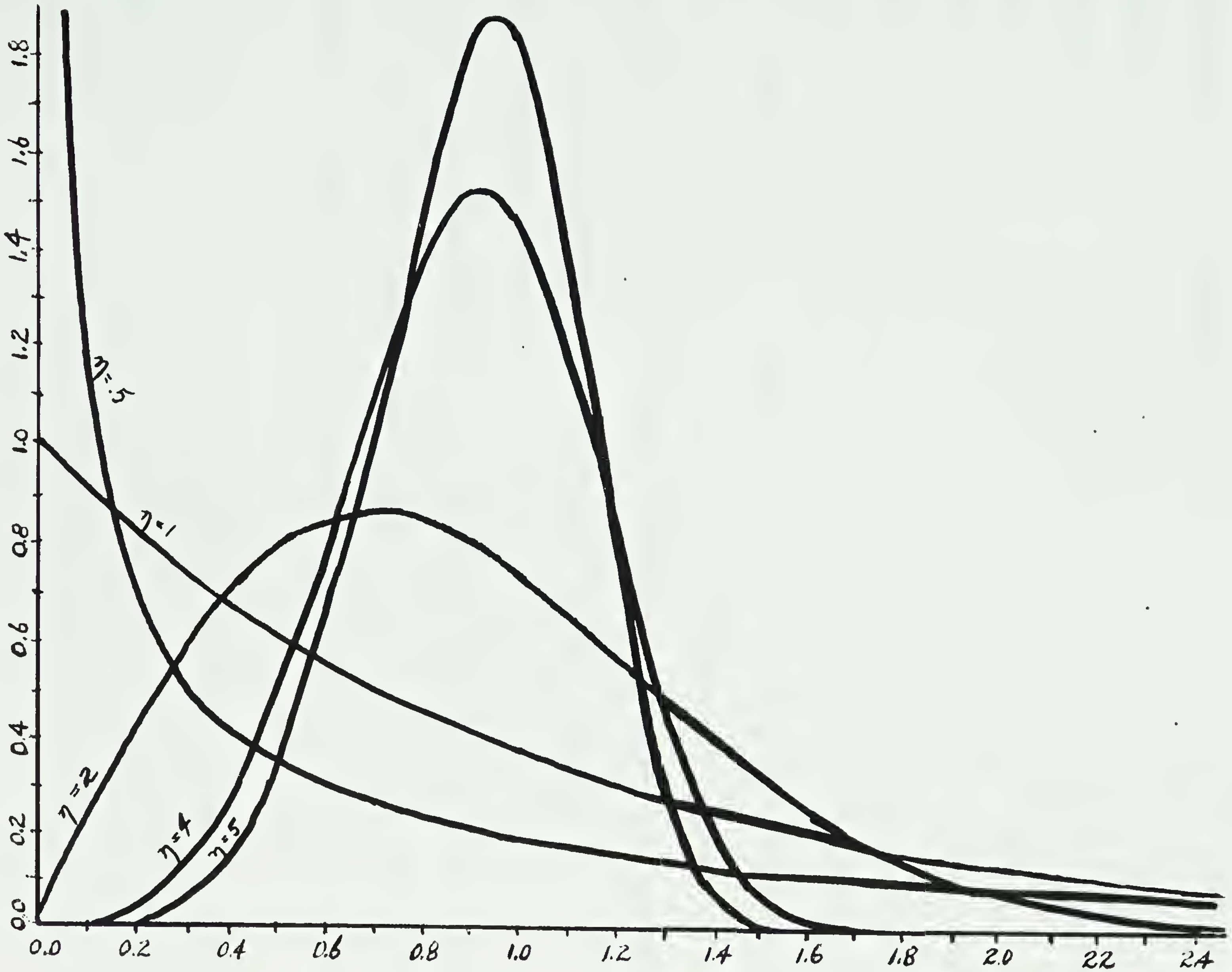
The random variable X is said to have the three-parameter Weibull distribution if the probability density function of X is

$$f(x;\theta,\eta,\tau) = \begin{cases} \frac{\eta}{\theta} \left(\frac{x-\tau}{\theta}\right)^{\eta-1} \exp -\left(\frac{x-\tau}{\theta}\right)^{\eta} , & x > \tau,\ \eta > 0,\ \theta > 0,\ \tau \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

giving a d.f. of $F(x;\theta,\eta,\tau) = 1- \exp\left\{-\left(\frac{x-\tau}{\theta}\right)^{\eta}\right\}$, and hazard rate (intensity) of $h(x;\theta,\eta,\tau) = \dfrac{\eta(x-\tau)^{\eta-1}}{\theta^{\eta}}$. In this case we write $X \sim WEI(\theta,\eta,\tau)$.

Figure 5.1.1 illustrates the shape of the probability density function $f(x;1,\eta,0)$ for various values of η.

Fig.5.1.1



Weibull p.d.f. for various values of η (θ =1, τ =0)

The Weibull distribution has many applications in life-testing or in problems where a skewed distribution is required. It may be considered to be a generalization of the gamma distribution to allow the hazard rate to depend on a power of X. In contrast to the exponential distribution with constant hazard (failure) rate $h(x;\theta) = \frac{1}{\theta}$ (i.e. $\eta = 1$), the Weibull distribution allows decreasing hazard rates (i.e. $\eta < 1$ implies work-hardened materials) or increasing hazard rates (i.e. $\eta > 1$ implies wearout). Many items, especially nonelectronic parts, exhibit increasing failure rates. Leiblein and Zelen (1956) used it to model ball-bearing failures; Kao (1959) used it to model vacuum-tube failure. Weibull (1951) derived it in the analysis of breaking strengths. Whatever the underlying distribution for positive random variables, the Weibull distribution is one of only two possible limit laws for $\min(X_1,\ldots,X_n)$ as $n \to \infty$. As such it is used to model metal fatigue breaking strength where strength is that at the weakest flaw (e.g. breaking strength of chain, ceramics, lumber, concrete, aircraft parts, etc.). In many applications the location parameter τ is known and without loss of generality we may take $\tau = 0$.

There exists a relationship between $WEI(\theta,\eta,0)$ and the Extreme-Value distribution $EV_I(\xi,b)$ given by $F(y) = 1\text{-}\exp\{-\frac{\exp(y-\xi)}{b}\}$, $-\infty < y < \infty$, $-\infty < \xi < \infty$, $b > 0$. If $X \sim WEI(\theta,\eta,0)$ then

$Y = \ell n\ X \sim EV_I(\xi = \ell n\ \theta,\ b = 1/\eta)$. Thus the Weibull distribution competing with the lognormal distribution is similar to the EV_I distribution competing with the normal distribution. For $Y \sim EV_I(\xi,b)$, $E(Y) = \xi - b\gamma$ and $Var(Y) = \frac{\pi^2 b^2}{6}$ where $\gamma = .5772$. (Euler's constant).

For known shape parameter η, we may use the transformation $Y = X^\eta$ to obtain $Y \sim EXP(\theta^\eta) = GAM(\theta^\eta,1,0)$. Outliers here may be handled using methods for the single parameter exponential family (see Chapter II).

5.2. Outlier-proneness of the Weibull and related models

First, let us restrict ourselves to the standard subfamily indexed only by shape parameter η. The probability density function is

$$f(x;\eta) = \begin{cases} \eta x^{\eta-1} e^{-x^{\eta}} & , \quad x > 0 \ , \quad \eta > 0 \\ 0 & , \quad \text{otherwise} \end{cases} .$$

The corresponding distribution function (d.f.) will be denoted as $F(x;\eta)$ and the family of such distribution functions as $\mathscr{F}$. In this model we consider i.i.d. random variables $X_1,\ldots,X_n$ with order statistics $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$.

Theorem 5.2.1: For i.i.d. random variables the family of Weibull distributions is outlier-prone on the right.

Proof; In order to prove Theorem 5.2.1 we notice that

$$P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)}) \cap X_{(1)} > 0\}$$

$$= P\{X_{(1)} > 0\}\cdot P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)})|X_{(1)} > 0\}$$

$$= P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)})\}P\{X_{(1)} > 0|X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)})\} .$$

Now $P\{X_{(1)} > 0\} = 1$ and

$$P\{X_{(1)} > 0|X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)})\} = 1$$

so that, following Neyman and Scott (1971),

$$P(k,n|\eta) = P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)})\} = P\{X_{(n)} > X_{(n-1)}$$

$$+ k(X_{(n-1)}-X_{(1)}) \mid X_{(1)} > 0\}$$

$$> P\{X_{(n)} > (k+1)X_{(n-1)}\} = Q(k,n|\eta) .$$

Thus it is sufficient to prove the stronger assertion that, as $\eta \to 0$, $Q(k,n|\eta) \to 1$.

$$Q(k,n|\eta) = n \int_0^\infty F^{n-1}\left(\frac{x}{k+1} ; \eta\right) f(x;\eta)dx$$

$$= n\int_0^\infty \{1 - e^{-\left(\frac{x}{k+1}\right)^\eta}\}^{n-1} \eta x^{\eta-1} e^{-x^\eta} dx .$$

Setting $u = 1 - e^{-\left(\frac{x}{k+1}\right)^\eta}$ we obtain

$$Q(k,n|\eta) = n(k+1)^\eta \int_0^1 u^{n-1}(1-u)^{(k+1)^\eta-1} du$$

$$= \frac{n\Gamma(n)\Gamma\{(k+n)^\eta\}(k+1)^\eta}{\Gamma\{n+(k+1)^\eta\}} .$$

It then follows that, as $\eta \to 0$, $Q(k,n|\eta) \to 1$ and thus $P(k,n|\eta) \to 1$ and $\Pi_1(k,n|\mathcal{F}) = 1$. Since this result holds for all $k > 0$, $n > 2$, the family of Weibull distributions is outlier-prone on the right.

Consider now the exchangeable model with (at most) one outlier observation. Kale (1975b) has shown that the exchangeable model based on scale parameter families for non-negative random variables with a possible change in scale is outlier-prone completely on the right. This would apply to the exchangeable model based on the Weibull distribution with possible change in the scale parameter θ. Let us consider the other possibility - the exchangeable model involving the Weibull distribution with possible change in the shape parameter η. We shall show that this model is outlier-resistant completely on the right.

Consider the situation where $n-1$ observations are from $WEI(1,\eta,0)$ and one observation is from $WEI(1,k^*\eta,0)$, $0 < k^* \leq 1$ and, a priori, each observation is equally likely to be the spurious one. Then $f(x;\underline{\theta})$ is $WEI(1,\eta,0)$ and $f(x;\underline{\xi})$ is $WEI(1,k^*\eta,0)$ and the likelihood may be written as

$$L(\underline{x};\eta,k^*) = \frac{1}{n} \sum_{r=1}^{n} \prod_{i \neq r} f(x_i;\underline{\theta}) f(x_r;\underline{\xi}),\ x_i > 0,\ 0 < k^* \leq 1$$

($k^* \leq 1$ since we are considering $X_{(1)}$ as a possible outlier). Then the joint density of the order statistics may be written as

$$f(x_{(1)},\ldots,x_{(n)}) = \frac{1}{n} \cdot n! \sum_{r=1}^{n} \frac{f(x_r;\eta k^*)}{f(x_r;\eta)} \prod_{i=1}^{n} f(x_i;\eta)$$

and

$g(x_{(1)},x_{(n-1)},x_{(n)})$

$$= \int_{x_{(1)}}^{x_{(n-1)}} \int_{x_{(1)}}^{x_{(n-2)}} \cdots \int_{x_{(1)}}^{x_{(3)}} f(x_{(1)},x_{(2)},\ldots,x_{(n)})dx_{(2)}\cdots dx_{(n-2)}$$

$$= (n-1)! \sum_{r=1}^{n} h_r(x_{(1)},x_{(n-1)},x_{(n)})$$

where

$$h_r(x_{(1)},x_{(n-1)},x_{(n)}) = \int_{S_{n-3}} \frac{f(x_r;\eta_1)}{f(x_r;\eta)} \prod_{i=1}^{n} f(x_i;\eta)dx_{(2)}\cdots dx_{(n-2)}$$

and S_{n-3} is the region $x_{(1)} < x_{(2)} < \cdots < x_{(n-2)} < x_{(n-1)}$.
For $r = 2,\ldots,n-2$

$$h_r(x_{(1)},x_{(n-1)},x_{(n)}) = \frac{f(x_{(1)};\eta)f(x_{(n-1)};\eta)f(x_{(n)};\eta)}{(r-2)!(n-r-2)!}$$

$$\int_{x_{(1)}}^{x_{(n-1)}} \{F(x_{(r)};\eta)-F(x_{(1)};\eta)\}^{r-2}\{F(x_{(n-1)};\eta)-F(x_{(r)};\eta)\}^{n-r-2}$$

$$f(x_{(r)};\eta_1)dx_{(r)}$$

while

$h_1(x_{(1)},x_{(n-1)},x_{(n)})$

$$= f(x_{(1)};\eta_1)f(x_{(n-1)};\eta)f(x_{(n)};\eta)\ \frac{[F(x_{(n-1)};\eta)-F(x_{(1)};\eta)]^{n-3}}{(n-3)!}$$

and

$h_{n-1}(x_{(1)},x_{(n-1)},x_{(n)})$

$$= f(x_{(1)};\eta)f(x_{(n-1)};\eta_1)f(x_{(n)};\eta)\ \frac{[F(x_{(n-1)};\eta)-F(x_{(1)};\eta)]^{n-3}}{(n-3)!}$$

and

$h_n(x_{(1)},x_{(n-1)},x_{(n)})$

$$= f(x_{(1)};\eta)f(x_{(n-1)};\eta)f(x_{(n)};\eta_1)\ \frac{[F(x_{(n-1)};\eta)-F(x_{(1)};\eta)]^{n-3}}{(n-3)!}$$

Letting $t(y) = f(y;\eta)$ and $s(y) = f(y;\eta_1)$ we obtain again expression (4.2.2) and, setting

$$p = P\{X_{(r)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

$$= \int_0^\infty \int_0^{x_{(n-1)}} \int_{(k+1)x_{(n-1)}-kx_{(1)}}^\infty g(x_{(1)},x_{(n-1)},x_{(n)})dx_{(n)}dx_{(1)}dx_{(n-1)}$$

and $x = x_{(1)}$ and $y = x_{(n-1)}$, from Theorem 4.2.1

$$p \leq 1 - k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

where S and s are, respectively, the distribution function and probability density function of the spurious observation and T and t are, respectively, the distribution function and probability density function of the non-spurious observations.

Theorem 5.2.2: The exchangeable model with (at most) one spurious observation based on the Weibull family of distributions indexed by shape parameter η is outlier-resistant completely on the right.

Proof: From Theorem 4.2.1 we know that if

$$p = P\{X_{(r)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

then, for non-negative random variables

$$p \leq 1 - k(n-1)(n-2) \int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

where $x = x_{(1)}$, $y = x_{(n-1)}$, S and s are, respectively, the distribution function and p.d.f. of a spurious observation and T and t are, respectively, the distribution function and p.d.f. of a non-spurious observation.

If, for all $k > 0$, $n > 2$ sup $p < 1$ then the family is outlier-resistant completely on the right. It is then sufficient to show that

$$(5.2.1)\quad J = k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$> 0.$$

Now

$$t(y) = \begin{cases} \eta y^{\eta-1}e^{-y^\eta}, & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$s(x) = \begin{cases} \eta_1 x^{\eta_1-1} e^{-x^{\eta_1}}, & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$T(b) = \begin{cases} 1-e^{-b^\eta}, & b > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$S(b) = \begin{cases} 1-e^{-b^{\eta_1}}, & b > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Then the left hand side of expression (5.2.1) becomes

$$(5.2.2)\quad J = k(n-1)(n-2)\int_0^\infty \eta y^{\eta-1}e^{-y^\eta}\int_0^y (1-e^{-x^{\eta_1}})\{(1-e^{-y^\eta})-(1-e^{-x^\eta})\}^{n-3}$$

(continued)

$$\times\ \eta\{(k+1)y-kx\}^{\eta-1} e^{-[(k+1)y-kx]^{\eta}} dxdy.$$

$$J \geq k(n-1)(n-2)\int_3^5 \eta y^{\eta-1} e^{-y^{\eta}} \int_1^2 (1-e^{-x^{\eta_1}})\{(1-e^{-y^{\eta}})-(1-e^{-x^{\eta}})\}^{n-3}$$

$$\eta\{(k+1)y-kx\}^{\eta-1} e^{-[(k+1)y-kx]^{\eta}} dxdy.$$

Now for $\eta > 0,\ k > 0,\ \eta_1 > 0,\ 3 \leq y \leq 5,\ 1 \leq x \leq 2$

$$y^{\eta-1} = \frac{y^{\eta}}{y} \geq \frac{3^{\eta}}{5} > 0$$

$$e^{-y^{\eta}} \geq e^{-5^{\eta}} > 0$$

$$(1-e^{-x^{\eta_1}}) \geq (1-e^{-1^{\eta_1}}) = 1 - \frac{1}{e} > 0 \quad \text{independently of} \quad \eta_1 > 0.$$

$$\{(1-e^{-y^{\eta}}) - (1-e^{-x^{\eta}})\}^{n-3} = \{e^{-x^{\eta}} - e^{-y^{\eta}}\}^{n-3} \geq \{e^{-2^{\eta}} - e^{-3^{\eta}}\}^{n-3} > 0$$

$$\{(k+1)y - kx\}^{\eta-1} = \frac{\{(k+1)y-kx\}^{\eta}}{\{(k+1)y-kx\}} \geq \frac{\{(k+1)3-2k\}^{\eta}}{\{(k+1)5-k\}} = \frac{(k+3)^{\eta}}{\{4k+5\}} > 0$$

$$e^{-[(k+1)y-kx]^{\eta}} \geq e^{-[(k+1)5-k]^{\eta}} = e^{-[4k+5]^{\eta}} > 0\ .$$

Therefore

$$J > k\eta^2(n-1)(n-2)\ \frac{3^{\eta}}{5}\ e^{-5^{\eta}}(1-\frac{1}{e})\{e^{-2^{\eta}}-e^{-3^{\eta}}\}^{n-3}\ \frac{(k+3)^{\eta}}{4k+5}$$

$$\times\ e^{-[4k+5]^{\eta}}{}_2$$

$$> 0 \quad \text{for all } k > 0,\ n > 2\ ,\ \eta,\ \eta_1 > 0.$$

Thus the exchangeable model with (at most) one spurious observation based on the Weibull family of distributions indexed by shape parameter η is outlier-resistant completely on the right.

5.3 Detection of Outliers

Consider a situation where $n-1$ observations are distributed as $WEI(1,\eta,0)$ and one is distributed as $WEI(1,k^*\eta,0)$, $0 < k \leq 1$ and we have the exchangeable model. If $k^* < 1$ we have exactly one spurious observation. Then $\Psi(x) = \frac{dG}{dF} = k^* x^{k^*\eta-\eta} e^{-(x^{k^*\eta}-x^{\eta})}$ is not monotone. Letting $u(r;n,\eta,k^*) = P(X_{(r)}$ is the spurious observation), we have

$$u(r;n,\eta,k^*) = \binom{n-1}{r-1}\int_0^\infty (1-e^{-x^\eta})^{r-1}(e^{-x^\eta})^{n-r} k^*\eta x^{k^*\eta-1} e^{-x^{k^*\eta}} dx$$

and, setting $y = e^{-x^{k^*\eta}}$, we obtain

$$u(r;n,k^*) = \binom{n-1}{r-1}\int_0^1 \left(1-e^{-(\ln 1/y)^{1/k^*}}\right)^{r-1}\left(e^{-(\ln 1/y)^{1/k^*}}\right)^{n-r} dy$$

which is independent of η. Therefore, without loss of generality (w.l.o.g.), we may take $\eta = 1/k^*$, giving

$$u(r;n,k^*) = \binom{n-1}{r-1}\int_0^\infty \left(1-e^{-x^{1/k^*}}\right)^{r-1}\left(e^{-x^{1/k^*}}\right)^{n-r} e^{-x} dx$$

and in particular

$$u(1;n,k^*) = \int_0^\infty \left(e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx$$

$$= \frac{1}{(n-1)^{k*}} \int_0^\infty e^{-v^{1/k*}} e^{-v/(n-1)^{k*}} dv$$

$$= \frac{1}{(n-1)^{k*}} L\left(u = 1/k*,\ p = \frac{1}{(n-1)^{k*}}\right)$$

where $L(u,p) = \int_0^\infty e^{-v^k} e^{-pv} dv$ is the Laplace transform of e^{-v^u} (Re $u > 0$).

Now $u(1;n,k*) = \int_0^\infty \left(e^{-x^{1/k*}}\right)^{n-1} e^{-x} dx \quad , \quad n > 1$

$$= \sum_{m=0}^\infty \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-(n-1)x^{1/k*}} dx$$

and setting $t = (n-1)x^{1/k*}$ we obtain

$$u(1;n,k*) = \sum_{m=0}^\infty \frac{(-1)^m}{m!} \int_{t=0}^\infty \frac{e^{-t} k* t^{k*(m+1)-1}}{(n-1)^{k*(m+1)}} dt$$

$$= \sum_{m=0}^\infty \frac{(-1)^m k* \Gamma\{k*(m+1)\}}{m!(n-1)^{k*(m+1)}} , \quad n > 1$$

(5.3.1) $$= \sum_{m=0}^\infty \frac{(-1)^m \Gamma\{k*(m+1)+1\}}{(m+1)!(n-1)^{k*(m+1)}} \quad , \ n > 1$$

For computational accuracy to the third decimal place, we would require

q terms where q is chosen such that $\frac{\Gamma\{k^*(q+1)+1\}}{(q+1)!(n-1)^{k^*(q+1)}} \leq 0.0005.$ (Apostol 1964). For fixed n, as $k^* \to \infty$ (heterogeneity increases) q increases; as $k^* \to 1$ (heterogeneity decreases) q decreases. Table 5.3.1 shows q values for some selected (n,k^*) values.

Table 5.3.1 Number of terms required to calculate $u(1;n,k^*)$ accurate to third decimal

		Sample size n			
		5	10	20	50
Coefficient	1/4	5	5	4	4
of spuriosity	1/2	5	4	3	3
k^*	3/4	5	3	3	2
	1	5	3	2	1

We can also develop a recursive formula for $u(r;n,k^*)$ in terms of $u(1;j,k^*)$, $j \leq n$. From equation 5.3.1 we have

$$u(1;n,k^*) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-1)^{k^*(m+1)}} \quad , \quad n > 1$$

and, for the case of $n = 1$, we may define $u(1;1,k^*) = 1$. For the special case of $r = n$ we may write

$$u(n;n,k^*) = \int_0^{\infty} \left(1-e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx$$

$$= \int_0^\infty \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j (e^{-x^{1/k^*}})^j e^{-x} dx$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \int_0^\infty e^{-jx^{1/k^*}} e^{-x} dx$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-jx^{1/k^*}} dx$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{k^*\Gamma\{k^*(m+1)\}}{j^{k^*(m+1)}}$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \frac{\Gamma\{k^*(m+1)+1\}}{j^{k^*(m+1)}}$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j u(1;j+1,k^*)$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j u(1;j+1,k^*)$$ where $u(1;1,k^*) = 1$ by definition.

For $r = 2,3,\ldots,n-1$

$$u(r;n,k^*) = \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \int_0^\infty (e^{-x^{1/k^*}})^{n-r+j} e^{-x}\, dx$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-x^{1/k^*}(n-r+j)} dx$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \left[\sum_{n=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-r+j)^{k^*(m+1)}} \right]$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j u(1;n-r+j+1,k^*) \ .$$

Thus we may write

$$u(r;n,k^*) = \begin{cases} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-1)^{k^*(m+1)}} & \text{for } r = 1,\ n \geq 2 \\ 1 \quad \text{if} \quad r = 1,\ n = 1 & \\ \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j u(1;n-r+j+1,k^*) & \text{for } r = 2,\ldots,n \\ 0 & \text{otherwise} \end{cases}$$

(See Appendix IV). Consider now

$$\lim_{k^*\to 1} u(r;n,k^*) = \begin{cases} \int_0^{\infty} \lim_{k^*\to 1} \left(e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx \ , & r = 1 \\ \int_0^{\infty} \lim_{k^*\to 1} \left(1-e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx \ , & r = n \\ \binom{n-1}{r-1} \int_0^{\infty} \lim_{k^*\to 1} \left(1-e^{-x^{1/k^*}}\right)^{r-1} \left(e^{-x^{1/k^*}}\right)^{n-r} e^{-x} dx \ , & r = 2,\ldots,n-1 \\ 0 & , \ \text{otherwise} \end{cases} .$$

Thus we find that

$$\lim_{k^*\to 1} u(r;n,k^*) = \begin{cases} \int_0^\infty e^{-xn}dx & , \ r = 1 \\ \int_0^\infty (1-e^{-x})^{n-1}e^{-x}dx & , \ r = n \\ \binom{n-1}{r-1}\int_0^\infty (1-e^{-x})^{r-1}e^{-x(n-r+1)}dx & , \ r = 2,\ldots,n-1 \\ 0 & , \ \text{otherwise} \end{cases}$$

$$= \begin{cases} 1/n & , \ r = 1 \\ 1/n & , \ r = n \\ \binom{n-1}{r-1}\dfrac{\Gamma(r)\Gamma(n-r+1)}{\Gamma(n+1} = 1/n & , \ r = 2,3,\ldots,n-1 \\ 0 & , \ \text{otherwise} \end{cases} .$$

The interpretation of this result is that, as $k^* \to 1$ ($k^* = 1$ is the case of homogeneous data), the heterogeneity of the data is decreasing and the probability of any order statistic being spurious approaches $1/n$, i.e. all equally likely.

Now consider $\lim_{k^*\to 0} u(r;n,k^*)$. We have

$$\lim_{k^*\to 0} u(r;n,k^*) = \begin{cases} \int_0^\infty \lim_{k^*\to 0} \left(e^{-x^{1/k^*}}\right)^{n-1}e^{-x}dx & , \ r = 1 \\ \int_0^\infty \lim_{k^*\to 0} \left(1-e^{-x^{1/k^*}}\right)^{n-1}e^{-x}dx & , \ r = n \\ \binom{n-1}{r-1}\int_0^\infty \lim_{k^*\to 0} \left(1-e^{-x^{1/k^*}}\right)^{r-1}\left(e^{-x^{1/k^*}}\right)^{n-r}e^{-x}dx & , \\ & \quad r = 2,\ldots,n-1 \\ 0 & , \ \text{otherwise} \end{cases}$$

Now

$$\lim_{k^*\to 0} u(1;n,k^*) = \lim_{k^*\to 0} \int_0^\infty \left(e^{-x^{1/k^*}}\right)^{n-1} e^{-x}dx$$

$$= \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m \left(e^{-x^{1/k^*}}\right)^{n-1} dx$$

$$= \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)\}}{m!\left|\frac{1}{k^*}\right|(n-1)^{k^*(m+1)}} \quad , \quad n > 1$$

$$= \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m k^*(m+1)\Gamma\{k^*(m+1)\}}{(m+1)!(n-1)^{k^*(m+1)}}$$

$$= \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-1)^{k^*(m+1)}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} = \frac{(-1)^0}{1!} + \frac{(-1)^1}{2!} + \frac{(-1)^3}{3!} + \ldots$$

$$= 1 - \frac{1}{e} \doteq .63 \ .$$

For $r = n$

$$\lim_{k^*\to 0} u(n;n,k^*) = \lim_{k^*\to 0} \int_0^\infty \left(1-e^{-x^{1/k^*}}\right)^{n-1} e^{-x}dx$$

$$= \lim_{k^*\to 0} \int_0^\infty \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j \left(e^{-x^{1/k^*}}\right)^j e^{-x}dx$$

$$= \lim_{k^*\to 0} \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j \int_0^\infty e^{-jx^{1/k^*}} e^{-x}dx$$

$$= \left\{ \lim_{k^*\to 0} \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \int_0^\infty e^{-jx^{1/k^*}} e^{-x}dx \right\} + \int_0^\infty e^{-x}dx$$

$$= \left\{ \lim_{k^*\to 0} \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \left[\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!\, j^{k^*(m+1)}} \right] \right\} + 1$$

$$= \left\{ \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \left[\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(1)}{(m+1)!} \right] \right\} + 1$$

$$= \left\{ \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j (1-\tfrac{1}{e}) \right\} + 1$$

$$= (1-\tfrac{1}{e}) \left\{ \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \right\} + 1$$

$$= (1-\tfrac{1}{e}) \left\{ \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j - \binom{n-1}{0}(-1)^0 \right\} + 1$$

$$= (1-\tfrac{1}{e})(-1) \quad + 1$$

$$= \frac{1}{e} \doteq .37 \quad .$$

For $r = 2,3,...,n-1$

$$\lim_{k^*\to 0} u(r;n,k^*) = \lim_{k^*\to 0} \binom{n-1}{r-1} \int_0^\infty \left(1-e^{-x^{1/k^*}}\right)^{r-1} \left(e^{-x^{1/k^*}}\right)^{n-r} e^{-x}dx$$

$$= \lim_{k^*\to 0} \binom{n-1}{r-1} \int_0^\infty \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \left(e^{-x^{1/k^*}}\right)^{j+n-r} e^{-x}dx$$

$$= \lim_{k^*\to 0} \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j$$

$$\left\{\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-(j+n-r)x^{1/k^*}} dx\right\}$$

$$= \lim_{k^*\to 0} \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j$$

$$\left\{\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)\}}{m! \left|\frac{1}{k^*}\right| (j+n-r)^{k^*(m+1)}}\right\}$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \lim_{k^*\to 0} \left\{\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(j+n-r)^{k^*(m+1)}}\right\}$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \left\{\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!}\right\}$$

$$= \binom{n-1}{r-1}\left(1-\frac{1}{e}\right)\left\{\sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j\right\}$$

$$= \binom{n-1}{r-1}\left(1-\frac{1}{e}\right)(-1+1)$$

$$= 0 \ .$$

Thus, as $k^* \to 0$ (i.e. one observation with Weibull shape parameter much smaller than the rest), we have

$$\lim_{k^* \to 0} u(r;n,k^*) = \begin{cases} 1-\frac{1}{e} & , \quad r=1 \\ \frac{1}{e} & , \quad r=n \\ 0 & , \quad \text{otherwise} \end{cases}$$

In this situation, heterogeneity is more evident and the probability that the first order statistic is spurious approaches $1 - \frac{1}{e} = .63$, the probability that the last order statistic is spurious approaches $\frac{1}{e} = .37$, and the probability that any other statistic is spurious approaches zero.

5.4 Estimation for Weibull parameters

5.4.1 Standard Estimators

Complete sufficient statistics do not exist for the Weibull distribution. Much of the estimation for the Weibull distribution falls into one of three categories:

i) For M.L.E's, the distributional results are not mathematically tractable hence Monte Carlo simulation is needed to tabulate percentage points. While the M.L.E.'s have optimality properties as n increases, they are not in closed form and require computer solution to obtain

$$\hat{\eta} = \left\{ \frac{\sum_{i=1}^{n} X_i^{\eta} \ell n\, x_i}{\sum_{i=1}^{n} X_i^{\eta}} - \frac{1}{n} \sum_{i=1}^{n} \ell n\, X_i \right\}^{-1} \quad \text{and} \quad \hat{\theta}^{\hat{\eta}} = \frac{\sum_{i=1}^{n} X_i^{\hat{\eta}}}{n} .$$

For $n \geq 100$, $\hat{\eta}$ is asymptotically $N\left(\eta, \frac{.608\eta^2}{n}\right)$ [see Cohen (1965) and Harter and Moore (1965)]. Inference procedures based on M.L.E.'s depend on pivotal quantities $\frac{\hat{\eta}}{\eta}$ and $\hat{\eta}\ \ell n\left(\frac{\hat{\theta}}{\theta}\right)$ [see Thoman, Bain and Antle (1970)].

ii) Best Linear Estimators have properties similar to M.L.E.'s. Mann and Fertig (1973) considered BLIE for the parameter b of EV_I. The BLIE $\tilde{b}_{BLIE} = \sum_{i=1}^{r} c_{i,r,n} X_{(i)}$ is a linear function of the BLUE and a maximal invariant since $\sum_{i=1}^{r} c_{i,r,n} = 0$. (r is the size of the

censored sample). The distribution of $\frac{\tilde{b}_{BLIE}}{b}$ is independent of b and ξ. Mann and Fertig (1977) also took Hassanein's (1972) asymptotically unbiased optimum estimators $\hat{b}_k = \sum_{i=1}^{k} b_{i,k} X_{(n_i)}$, $n_i = [\gamma_{i,k} n] + 1$ and obtained, for small samples, the unbiased

$$b^* = \frac{\hat{b}_k}{\bar{b}_{k,n}}$$

where $\bar{b}_{k,n} = E\left[\sum_{i=1}^{k} b_{i,k} Z_{(n_i)}\right]$, where $Z_i \sim EV_I(0,1)$ and $\frac{2b^*}{cb} \stackrel{\bullet}{\sim} \chi^2_{2/c}\,df$ and the BLIE $b^*_{BLIE} = \frac{b^*}{1+c_{k,n}}$ where $c_{k,n} = Var\left(\frac{b^*}{b}\right)$.

iii) Simple estimators, such as good linear unbiased estimators (GLUE's) and modified GLUE's, are identical to BLUE's if $r = 2$ and similar but not identical if $r > 2$, where r is the size of the censored sample. They are essentially equivalent to M.L.E.'s for censored sampling but are slightly less efficient for the complete sample case. Bain (1973) suggested using the approximate BLUE

$$\hat{b}_B = \frac{\sum_{i=1}^{r} |y_{(i)} - y_{(r)}|}{nk_{r,n}} = \frac{(r-1)y_{(r)} - \sum_{i=1}^{r-1} y_{(i)}}{nk_{r,n}}$$

where $y_i \sim EV_I(\xi, b)$ and $k_{r,n} = -\frac{1}{n} E\left\{\sum_{i=1}^{n-1} (w_i - w_r)\right\}$ where w_i are

the order statistics of $EV_I(0,1)$. Then $2nk_{r,n} \frac{\hat{b}_B}{b} \sim \chi^2_{2nk_{r,n}}$ df (if $r/n < 1/2$) and $\hat{\eta}_B = \frac{nk_{r,n}-1}{nk_{r,n}\hat{b}_B}$ is approximately unbiased for η. For complete samples (i.e. $r = n$), $\hat{b}_B$ has zero asymptotic relative efficiency.

Englehardt and Bain (1973) suggested $\hat{b}_s = \dfrac{\sum_{i=1}^{r} |y_{(i)}-y_{(s)}|}{nk_{s,r,n}}$ as an improvement where

$$s = \begin{cases} n & 2 \leq n \leq 15 \\ n-1 & 16 \leq n \leq 24 \\ [.892\ n] + 1\ , & n \geq 25 \\ r\ , & r \leq .9n \end{cases} \quad \text{and } r > .9n \text{ (first three cases)}$$

and $h \frac{\hat{b}_s}{b} \overset{\cdot}{\sim} \chi^2_h$ df where $h = \dfrac{2}{\mathrm{Var}\left(\frac{\hat{b}_s}{b}\right)}$.

For $\frac{r}{n} \to 0$ (very heavy censoring) as $n \to \infty$, $\hat{b}_B$ and M.L.E. $\hat{b}$ appear to agree. Compared to $\hat{b}_B$, $\hat{b}_s$ has relative efficiency ranging from 1 (n=2) to 0.82 (n=25).

Mann and Fertig (1975) converted Bain's estimator to an approximate BLIE. For $n \geq 20$, $\dfrac{2\frac{\hat{b}_s}{b}}{\ell_{r,n}} \overset{\cdot}{\sim} \chi^2_{2/\ell_{r,n}}$ df where

$\hat{b}_s = \dfrac{\sum_{i=1}^{r} |y_{(s)}-y_{(i)}|}{nk_{s,r,n}}$ and approximate BLIE $\tilde{b}_{MF} = \dfrac{\hat{b}_s}{1+\ell_{r,n}}$, where

$\ell_{r,n} = \mathrm{Var}\left(\frac{\hat{b}_s}{b}\right) = \frac{1}{nk_{s,r,n}}$. Then $\mathrm{MSE}(\tilde{b}_{MF}) = \frac{\ell_{r,n} b^2}{1+\ell_{r,n}}$. For heavy censoring ($r << n$), $\tilde{b}_{MF}$ is closer to $\hat{b}_{MLE}$ then $\hat{b}_s$.

In terms of the Weibull shape parameter η, $a\hat{\eta} = a\left(\frac{1}{\tilde{b}_{MF}}\right)$ is unbiased for η if $a = \frac{h-2}{h+2}$ and $d\hat{\eta} = d\left(\frac{1}{\tilde{b}_{MF}}\right)$ has minimum MSE if $d = \frac{h-4}{h-2}$ where $\frac{h+2}{h} = 1 + \ell_{r,n}$ (i.e. $h = \frac{2}{\mathrm{Var}(\hat{b}_s/b)}$).

Englehardt and Bain (1977a) proposed, as a new simple unbiased estimator for complete samples to replace $\hat{b}_s$,

$$\hat{b}_{EB} = \frac{\left(-\sum_{i=1}^{s} y_{(i)} + \frac{s}{n-s}\sum_{i=s+1}^{n} y_{(i)}\right)}{nk_n}$$

where $s = [qn]$, $0 < q < 1$ is chosen to minimize the asymptotic variance of $\hat{b}_{EB}$. This results in

$$s = [.84n]$$

and

$$k_n = E\left[-\sum_{i=1}^{s} w_{(i)} + \frac{s}{n-s}\sum_{i=s+1}^{n} w_{(i)}\right]/n \ , \quad w_i \sim EV(0,1)$$

$$= -\left[E\sum_{i=1}^{s} w_{(i)} + s\gamma\right]/(n-s)$$

and γ is Euler's constant. Then

$$n\ \mathrm{Var}\left(\frac{\hat{b}_{EB}}{b}\right) = \frac{V_s + \left(\frac{s}{n-s}\right)U_{s+1} - \frac{s\pi^2}{6}}{(n-s)k_n^2}$$

where $V_s = \mathrm{Var}\left(\sum_{i=1}^{s} w_{(i)}\right)$, $U_{s+1} = \mathrm{Var}\left(\sum_{i=s+1}^{n} w_{(i)}\right)$, $\mathrm{Var}\left(\sum_{i=1}^{n} w_{(i)}\right) = \frac{n\pi^2}{6}$.

To obtain an approximate BLIE (or MLE) we may use $b^*_{EB} = \dfrac{\hat{b}_{EB}}{1+\mathrm{Var}\left(\frac{\hat{b}_{EB}}{b}\right)}$.

Three other simple estimators have been suggested. Menon (1963) used the transformation $Z = \ln\left(\frac{X}{\theta}\right)^{\eta}$ where $X \sim WEI(\theta,\eta,0)$ and $\ln X \sim EV_I(\xi,b)$. He suggested $\hat{b}_{MN} = \left(\frac{6s^2_{\ln x}}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{MN} = \widehat{\ln\theta}$ $= \frac{\sum_{i=1}^{n} \ln X_i}{n} + \gamma\hat{b}_M$. Then $\hat{\eta}_{MN} = \frac{1}{\hat{b}_M} \overset{\bullet}{\sim} N\left(\eta, \frac{1.1\eta^2}{n}\right)$ and $\hat{\theta}_{MN} = e^{\hat{\xi}_M}$ $\sim N\left(\theta, \frac{1.2b^2\theta^2}{n}\right)$. Dubey (1967) suggested estimating η by using percentiles. Based upon two percentiles,

$$\hat{\eta}_D = \frac{\ln\{-\ln(1-p_1)\}-\ln\{-\ln(1-p_2)\}}{\ln y_{p_1} - \ln y_{p_2}}$$

where $p_1 = 17\%$ and $p_2 = 97\%$ minimize $\mathrm{Var}(\hat{\eta}_D)$ and

$$y_p = \begin{cases} x_{(np)} & \text{if } np \text{ is an integer} \\ x_{[np]+1} & \text{if } np \text{ is not an integer.} \end{cases}$$

Asymptotically $\hat{\eta}_D \overset{\bullet}{\sim} N\left(\eta, \frac{\eta^2(.91627479)}{n}\right)$.

Murthy and Swartz (1975) used an approach similar to Dubey but used the EV_I distribution and obtained an estimator of b based upon two order statistics

$$\hat{b}_{MS} = \{\ell n\ T_{(j)} - \ell_n\ T_{(\ell)}\}\ B(N,\ell,j)$$

where $B(N,\ell,j) = \frac{1}{2E(Y)}$ and $Y = \frac{\ell n\ T_{(j)} - \ell n\ T_{(\ell)}}{2b}$. This is MVUE and has relative efficiency w.r.t. the Cramer-Rao lower bound (CRLB) approaching 70%. The following Table 5.4.1 gives the optimum values for j and ℓ.

Table 5.4.1

Optimal Order Statistics for Murthy-Swartz Estimator of $b = \frac{1}{\eta}$

n	j	ℓ
$2 \leq n \leq 5$	n	1
$6 \leq n \leq 10$	n	2
$11 \leq n \leq 16$	n	3
$17 \leq n \leq 23$	n	4
$24 \leq n \leq 26$	n	5

5.4.2 Estimators suggested for use.

Under the exchangeable model with a shape change, we assume $n-1$

observations are governed by $f(x;\theta,\eta) = \frac{\eta x^{\eta-1}}{\theta^{\eta}} \exp\{-(\frac{x}{\theta})^{\eta}\}$, $x > 0$, $\eta, \theta > 0$ and one observation is governed by $f(x;\theta,k^*\eta)$, $0 < k^* \leq 1$. If $0 < k^* < 1$, we have exactly one spurious observation. We are interested in estimating η (or in the EV_I case, $b = \frac{1}{\eta}$).

From Table 5.4.1, for $2 \leq n \leq 26$, the optimal form of the Murthy-Swartz estimator uses the largest order statistic. Dubey's estimator involves one or both of the smallest and largest order statistics for $2 \leq n \leq 33$. Also $\hat{b}_s$, $2 \leq n \leq 15$, $\hat{b}_B$ and $\hat{b}_{EB}$ all involve these two highly suspect values. Menon's estimator

$\hat{b}_{MN} = \sqrt{\frac{6s^2_{\ell n\ x}}{\pi^2}}$ has

$$MSE(\hat{b}_{MN}) = Var(\hat{b}_{MN}) + \{Bias(\hat{b}_{MN})\}^2$$

$$= \frac{1.1\ b^2}{n} + b^2 O(\frac{1}{n^2}) + \{bO(\frac{1}{n})\}^2$$

under the homogeneous model. However, with the above exchangeable model,

$$E_{het}(\hat{b}_{MN}) = \frac{\sqrt{6}}{\pi}\ \{\sqrt{Var_{\ell n\ x}} + bO(\frac{1}{n})\}\ \text{[see Cramer (1946), 27.7.1]}$$

$$= \frac{\sqrt{6}}{\pi}\ \{\sqrt{\frac{\pi^2}{6} b^2 (\frac{n-1}{n}) + \frac{\pi^2}{6}(\frac{b}{k^*})^2 \frac{1}{n}} + bO(\frac{1}{n})\}$$

$$= b\sqrt{\frac{n-(1-\frac{1}{k^{*2}})}{n}} + bO\left(\frac{1}{n}\right)$$

and as $k^* \to 0$, bias $\to \infty$.

Also $MSE_{het}(\hat{b}_{MN}) = Var_{het}(\hat{b}_{MN}) + \{Bias(\hat{b}_{MN})\}^2$ and

$$Var_{het}(\hat{b}_{MN}) = \frac{6}{\pi^2}\left\{\frac{\mu_4-\mu_2^2}{4n\mu_2} + O\left(\frac{1}{n^2}\right)\right\} \quad \text{[see Cramer (1946), 27.7.2]}$$

$$= \frac{6}{\pi^2}\left[\frac{3b^2}{2\pi^2}\left\{\frac{\pi^4}{15}\left(\frac{n-1+\frac{1}{k^{*4}}}{n}\right) + \frac{4(2.404)\gamma(n-1+\frac{1}{k^{*3}})(n-1+\frac{1}{k^*})}{n^2}\right.\right.$$

$$\left.\left. + \frac{\pi^2\gamma^2(n-1+\frac{1}{k^{*2}})(n-1+\frac{1}{k^*})^2}{n^3} - \frac{3\gamma^4(n-1+\frac{1}{k^*})^4}{n^4} - \frac{\pi^4(n-1+\frac{1}{k^*})^2}{36n^2}\right\} + b^2O\left(\frac{1}{n^2}\right)\right]$$

(see Appendix V).

Taking $\lim_{k^*\to 0} MSE_{het}(\hat{b}_{MN})$, we find $MSE(\hat{b}_{MN}) \to \infty$ as $k^* \to 0$. Thus this estimation technique seems poor if an outlier is present. Also $s^2_{\ell n\, x}$ would be strongly affected by the presence of an outlier occurring at $X_{(1)}$ or at $X_{(n)}$ and these are precisely the values where $u(r;n,k^*)$ is largest as $k^* \to 0$. We might obtain a more robust estimator by using trimming, winsorization or semi-winsorization. Thus we might consider

i) $\hat{b}_{RA} = \left(\frac{6\hat{\sigma}_A^2}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{RA} = \hat{\mu}_A + \gamma\hat{b}_{RA}$

ii) $\hat{b}_{RW} = \left(\frac{6\hat{\sigma}_W^2}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{RW} = \hat{\mu}_W + \gamma\hat{b}_{RW}$

iii) $\hat{b}_{RS} = \left(\frac{6\hat{\sigma}_S^2}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{RS} = \hat{\mu}_S + \gamma\hat{b}_{RS}$

where $Y_i = \ln X_i$, $b = \frac{1}{\eta}$, $\xi = \ln\theta$ and $\hat{\sigma}_A^2$, $\hat{\sigma}_W^2$, $\hat{\sigma}_S^2$, $\hat{\mu}_A$, $\hat{\mu}_W$ and $\hat{\mu}_S$ are as defined in Appendix I and γ is Euler's constant.

Table 5.4.2 Computation of means and variances and winsorized means and variances for 25 samples of size five based on $EV_I(0,1/\eta)$ with one spurious observation present (see Appendix I)

η = 5 00000 κ^* = 0 10000

Weibull (V)

0 102121E+01	0 881289E+00	0 125484E+01	0 812620E+00	0 222729E+01
0 803152E+00	0 937152E+00	0 805811E+00	0 590656E+00	0 370809E+01
0 862987E+00	0 100282E+01	0 123895E+01	0 841884E+00	0 423470E+00
0 937318E+00	0 105347E+01	0 757928E+00	0 750034E+00	0 423634E-01
0 827337E+00	0 877620E+00	0 762281E+00	0 123292E+01	0 127538E+01
0 881348E+00	0 856771E+00	0 100320E+01	0 107197E+01	0 841635E-01
0 820470E+00	0 108828E+01	0 120828E+01	0 889250E+00	0 493110E-02
0 105857E+01	0 887128E+00	0 867488E+00	0 848710E+00	0 387857E+00
0 138243E+01	0 725363E+00	0 825886E+00	0 678646E+00	0 359345E+00
0 813841E+00	0 107227E+01	0 764260E+00	0 936452E+00	0 387794E+00
0 837215E+00	0 885044E+00	0 548735E+00	0 310820E+00	0 134010E+01
0 105094E+01	0 815291E+00	0 121399E+01	0 110028E+01	0 590449E-02
0 624547E+00	0 931725E+00	0 121240E+01	0 110820E+01	0 252715E+01
0 885510E+00	0 760880E+00	0 122693E+01	0 101766E+01	0 577123E+00
0 909748E+00	0 880684E+00	0 703416E+00	0 113378E+01	0 902794E+00
0 548408E+00	0 881488E+00	0 115158E+01	0 138753E+01	0 116566E+00
0 752757E+00	0 888805E+00	0 107318E+01	0 115550E+01	0 111080E+01
0 468859E+00	0 841808E+00	0 706768E+00	0 840323E+00	0 222881E-01
0 116367E+01	0 727470E+00	0 124621E+01	0 564764E+00	0 340384E+00
0 125350E+01	0 113521E+01	0 399613E+00	0 874926E+00	0 916387E+00
0 739252E+00	0 882368E+00	0 894012E+00	0 104101E+01	0 495553E+01
0 104428E+01	0 106242E+01	0 984392E+00	0 765419E+00	0 344395E-01
0 104512E+01	0 116185E+01	0 101539E+01	0 100000E+01	0 281081E-01
0 764350E+00	0 923'14E+00	0 864647E+00	0 661185E+00	0 301828E-02
0 902853E+00	0 810298E+00	0 773216E+00	0 904088E+00	0 261505E-01

Extreme Value (W)

0 209857E-01	-0 188888E-01	0 227088E+00	-0 207491E+00	0 800785E+00
-0 219212E+00	-0 649100E-01	-0 989251E-01	-0 528522E+00	0 130888E+01
-0 377159E-01	0 261685E-02	0 215074E+00	-0 588728E-01	-0 858273E+00
-0 647313E-01	0 520921E-01	-0 277168E+00	-0 287636E+00	-0 318147E+01
-0 754381E-01	-0 130542E+00	-0 271440E+00	0 209382E+00	0 243242E+00
-0 413474E+00	-0 154584E+00	0 319303E-02	0 884987E-01	-0 247488E+01
-0 197878E+00	0 841838E-01	0 186202E+00	-0 108085E-01	-0 531218E+01
0 550264E-01	-0 108555E+00	-0 108158E+00	-0 164038E+00	-0 100008E+01
0 323842E+00	-0 321097E+00	-0 768980E-01	-0 385742E+00	-0 102347E+01
-0 488019E+00	0 697750E-01	-0 268847E+00	-0 656587E-01	-0 100023E+01
-0 648429E-01	-0 110882E+00	-0 598318E+00	-0 116854E+01	0 292745E+00
0 496879E-01	-0 885127E-01	0 193914E+00	0 655844E-01	-0 513204E+01
-0 470729E+00	-0 707178E-01	0 192588E+00	0 102741E+00	0 927094E+00
-0 145959E-01	-0 286510E+00	0 204514E+00	0 175057E-01	-0 548889E+00
-0 945874E-01	-0 150051E+00	-0 351806E+00	0 125558E+00	-0 102261E+00
-0 598914E+00	-0 864804E-02	0 141132E+00	0 313007E+00	-0 214930E+01
-0 284013E+00	-0 308602E-01	0 706295E-01	0 144531E+00	0 105167E+00
-0 757452E+00	-0 598498E-01	-0 347052E+00	-0 173888E+00	-0 380388E+01
0 151577E+00	-0 318183E+00	0 220108E+00	-0 571348E+00	-0 107788E+01
0 225939E+00	0 126819E+00	-0 917260E+00	-0 133615E+00	-0 873161E-01
-0 302117E+00	-0 177689E-01	-0 112037E+00	0 401839E-01	0 100050E+01
0 433347E-01	0 605533E-01	-0 157307E-01	-0 267331E+00	-0 336855E+01
0 441359E-01	0 150098E+00	0 152692E-01	0 324247E-05	-0 357170E+01
-0 268730E+00	-0 800028E-01	-0 145434E+00	-0 413722E+00	-0 580307E+01
-0 102195E+00	-0 210354E+00	-0 257197E+00	-0 100829E+00	-0 364389E+01

Sample	$\bar{W}$	s^2_W	$\bar{W}_{(2,1)}$	$s^2_{(2,1)W}$	$\bar{W}_{(5,1)}$	$s^2_{(5,1)W}$	$\bar{W}_{(4,5)}$	$s^2_{(4,5)W}$	$\bar{W}_{(1,5)}$	$s^2_{(1,5)W}$
1	0 16450	0 15035	0 20222	0 12238	0 36615	0 16614	0 04976	0 03365	-0 03716	0 03289
2	0 08008	0 50581	0 14154	0 43149	0 44738	0 62335	-0 19490	0 03839	-0 28722	0 05101
3	-0 14783	0 17008	0 01205	0 01353	0 06704	0 01876	-0 19033	0 14056	-0 36270	0 20599
4	-0 74778	1 84136	-0 17302	0 02488	-0 10507	0 02848	-0 77115	1 79737	-1 39049	2 62154
5	-0 00496	0 04982	0 02322	0 03502	0 09798	0 03423	-0 01173	0 04585	-0 10790	0 03893
6	-0 59407	1 14017	-0 18177	0 05137	-0 08517	0 04206	-0 60733	1 11905	-1 10297	1 59083
7	-1 05350	5 68732	-0 03064	0 02841	0 04678	0 02601	-1 07850	5 61277	-2 15378	8 32210
8	-0 26516	0 17550	-0 09795	0 00809	-0 05414	0 01045	-0 28779	0 15470	-0 47817	0 22923
9	-0 29667	0 24253	-0 16913	0.09212	-0 02721	0 11596	-0 37682	0 15031	-0 56614	0 18756
10	-0 35060	0 17637	-0 24815	0 06247	-0 13659	0 05780	-0 37768	0 15158	-0 56460	0 18046
11	-0 32997	0 32032	-0 21592	0 14626	-0 03771	0 13469	-0 40149	0 23456	-0 62222	0 29241
12	-0 97628	5 40733	0 03243	0 01490	0 08891	0 01378	-0 99595	5 35172	-2 04147	7 96429
13	0 13620	0 28025	0 21620	0 17086	0 41576	0 22684	-0 01070	0 07769	-0 14337	0 09826
14	-0 12576	0 08688	-0 07312	0 04495	0 02509	0 04078	-0 16316	0 06299	-0 27660	0 07610
15	-0 11463	0 02894	-0 07428	0 01315	-0 01916	0 01790	-0 15866	0 01219	-0 21010	0 01718
16	-0 46052	1 00894	-0 15045	0 18056	0 03194	0 14238	-0 49490	0 94837	-0 95298	1 26919
17	0 00113	0 02963	0 05180	0 00635	0 08684	0 00527	-0 00674	0 02712	-0 08468	0 03564
18	-1 02640	2 47696	-0 41916	0 10583	-0 27963	0 08516	-1 05122	2 42430	-1 77716	3 46716
19	-0 31911	0 28784	-0 21784	0 14707	-0 05955	0 13245	-0 33281	0 27030	-0 57866	0 27480
20	-0 15709	0 20265	-0 00036	0 02762	0 07155	0 02951	-0 17691	0 18563	-0 38573	0 24509
21	0 24176	0 59379	0 27977	0 54932	0 62228	0 80039	-0 07031	0 02066	-0 13877	0 02519
22	-0 70955	2 22678	-0 08930	0 02721	-0 02372	0 01953	-0 71299	2 22021	-1 39537	3 25816
23	-0 67244	2 63022	0 04190	0 00398	0 07192	0 00534	-0 69363	2 58888	-1 41680	3 86992
24	-1 34219	6 23479	-0 26432	0 02319	-0 19758	0 02054	-1 35528	6 19435	-2 48880	9 17369
25	-0 86289	2 42150	-0 18555	0 00625	-0 15423	0 00554	-0 86317	2 42098	-1 57150	3 58214

We could seek M.L.E's of θ and η under the exchangeable model where

$$L(\underline{x};k^*,\eta,\theta) = \frac{k^*\eta^n}{n\theta^{\eta(n+k^*-1)}} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\eta}} \cdot$$

$$\prod_{i=1}^{n} x_i^{\eta-1}\left\{\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}\right\}$$

and

$$K(\underline{x};k^*,\eta,\theta) = \ell n\ L(\underline{x};k^*,\eta,\theta)$$

$$= \ell n\ k^* + n\ \ell n\ \eta - \ell n\ n - \eta(n+k^*-1)\ell n\ \theta - \sum_{i=1}^{n}(x_i/\theta)^{\eta}$$

$$+ \sum_{i=1}^{n}(\eta-1)\ell n\ x_i + \ell n\left\{\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}\right\}.$$

Then

$$\frac{\partial K}{\partial \theta} = \frac{-\eta(n+k^*-1)}{\theta} + \sum_{i=1}^{n}\frac{\eta\ x_i^{\eta}}{\theta^{\eta+1}}$$

(continued)

$$+ \frac{\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}} \{k^*\eta \frac{x_i^{k^*\eta}}{\theta^{k^*\eta+1}} - \frac{\eta x_i^{\eta}}{\theta^{\eta+1}}\}}{\sum_{i=1}^{n} x_i^{\eta(k^*+1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}}$$

$$\frac{\partial K}{\partial \eta} = \frac{n}{\eta} - (n+k^*-1)\ln\theta - \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\eta} \ln\left(\frac{x_i}{\theta}\right) + \sum_{i=1}^{n} \ln x_i$$

$$+ \Big[\sum_{i=1}^{n} x_i^{\eta(k^*-1)} (\ln x_i)(k^*-1) e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}$$

$$+ x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}} \{-k^* \left(\frac{x_i}{\theta}\right)^{k^*\eta} \ln\left(\frac{x}{\theta}\right) + \left(\frac{x_i}{\theta}\right)^{\eta} \ln\left(\frac{x_i}{\theta}\right)\}\Big]$$

$$\Big[\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}\Big]^{-1}$$

$$\frac{\partial K}{\partial k^*} = \frac{1}{k^*} - \eta \ln\theta + \Big[\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}\Big]^{-1}$$

$$\Bigg[\sum_{i=1}^{n} \Big[x_i^{\eta(k^*-1)} (\ln x_i) \eta e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}$$

$$+ x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}} \{-\eta (x_i/\theta)^{k^*\eta} \ln\left(\frac{x_i}{\theta}\right)\}\Big]\Bigg]$$

The above equations appear mathematically intractable.

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Appendix I: The A-, W-, and S-Rules and Premium/Protection.

We adopt the following notation:

$$z_i = y_i - \bar{y} \qquad \bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \qquad s^2 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n-1}$$

$$\bar{y}_{(1)} = \frac{\sum_{i=2}^{n} y_{(i)}}{n-1} \qquad \bar{y}_{(n)} = \frac{\sum_{i=1}^{n-1} y_{(i)}}{n-1} \qquad \bar{y}_{(j,k)} = \frac{\left(\sum_{i=1}^{n} y_{(i)} + y_{(j)} - y_{(k)}\right)}{n}$$

$$s^2_{(1)} = \frac{\sum_{i=2}^{n} \left(y_{(i)} - \bar{y}_{(1)}\right)^2}{n-2} \qquad s^2_{(n)} = \frac{\sum_{i=1}^{n-1} \left(y_{(i)} - \bar{y}_{(n)}\right)^2}{n-2}$$

$$s^2_{(j,k)} = \frac{1}{(n-1)} \left\{ \sum_{i=1}^{n} \left(y_{(i)} - \bar{y}_{(j,k)}\right)^2 + \left(y_{(j)} - \bar{y}_{(j,k)}\right)^2 - \left(y_{(k)} - \bar{y}_{(j,k)}\right)^2 \right\}.$$

Anscombe's Rule (A-Rule):

$$\mu_A = \begin{cases} \bar{y} & \text{if } |z_{(n)}| < Cs \text{ and } |z_{(1)}| < Cs \\ \bar{y}_{(1)} & \text{if } |z_{(1)}| \geq Cs \text{ and } |z_{(1)}| > |z_{(n)}| \\ \bar{y}_{(n)} & \text{if } |z_{(n)}| \geq Cs \text{ and } |z_{(n)}| > |z_{(1)}| \end{cases} .$$

$$\sigma^2_A = \begin{cases} Ds^2 & \text{if } z^2_{(1)} < Ks^2 \text{ and } z^2_{(n)} < Ks^2 \\ Ds^2_{(1)} & \text{if } z^2_{(1)} \geq Ks^2 \text{ and } z^2_{(1)} > z^2_{(n)} \\ Ds^2_{(n)} & \text{if } z^2_{(n)} \geq Ks^2 \text{ and } z^2_{(n)} \geq z^2_{(1)} \end{cases} .$$

Winsorization (W-Rule):

$$\mu_W = \begin{cases} \bar{y} & \text{if } |z_{(1)}| < Cs \text{ and } |z_{(n)}| < Cs \\ \bar{y}_{(2,1)} & \text{if } |z_{(1)}| \geq Cs \text{ and } |z_{(1)}| > |z_{(n)}| \\ \bar{y}_{(n-1,n)} & \text{if } |z_{(n)}| \geq Cs \text{ and } |z_{(n)}| > |z_{(1)}| \end{cases} .$$

$$\sigma_W^2 = \begin{cases} Ds^2 & \text{if } z_{(1)}^2 < Ks^2 \text{ and } z_{(n)}^2 < Ks^2 \\ D \max\left[s_{(2,1)}^2, s_{(n,1)}^2\right] & \text{if } z_{(1)}^2 \geq Ks^2 \text{ and } z_{(1)}^2 > z_{(n)}^2 \\ D \max\left[s_{(1,n)}^2, s_{(n-1,n)}^2\right] & \text{if } z_{(n)}^2 \geq Ks^2 \text{ and } z_{(n)}^2 > z_{(1)}^2 \end{cases} .$$

Semi-Winsorization (S-Rule):

$$\mu_S = \begin{cases} \bar{y} & \text{if } |z_{(1)}| < Cs \text{ and } |z_{(n)}| < Cs \\ \dfrac{(n-1)\bar{y}_{(1)}+(\bar{y}-Cs)}{n} & \text{if } |z_{(1)}| \geq Cs \text{ and } |z_{(1)}| > |z_{(n)}| \\ \dfrac{(n-1)\bar{y}_{(n)}+(\bar{y}+Cs)}{n} & \text{if } |z_{(n)}| \geq Cs \text{ and } |z_{(n)}| > |z_{(1)}| . \end{cases}$$

$$\sigma_S^2 = \begin{cases} Ds^2 & \text{if } z_{(1)}^2 < Ks^2 \text{ and } z_{(n)}^2 < Ks^2 \\ \dfrac{D}{n-1}\left\{(n-2)s_{(1)}^2 + Ks^2\right\} & \text{if } z_{(1)}^2 \geq Ks^2 \text{ and } z_{(1)}^2 > z_{(n)}^2 \\ \dfrac{D}{n-1}\left\{(n-2)s_{(n)}^2 + Ks^2\right\} & \text{if } z_{(n)}^2 \geq Ks^2 \text{ and } z_{(n)}^2 > z_{(1)}^2 \end{cases} .$$

If we adopt the premium-protection approach suggested by Anscombe (1960) we define

$$\text{Premium} = \frac{\text{MSE(new estimator)} - \text{MSE(old estimator)}}{\text{MSE(old estimator)}}$$

assuming homogeneous data

and

$$\text{Protection} = \frac{\text{MSE(old estimator)} - \text{MSE(new estimator)}}{\text{MSE(old estimator)}}$$

assuming spurious values(s) are present.

Appendix II: Asymptotic approximation for $\Gamma(\nu+1)$

Consider $\Gamma(\nu+1) = \int_0^\infty e^{-u}u^\nu\, du$. Setting $u = \nu t$, $du = \nu dt$ and

$\Gamma(\nu+1) = \int_0^\infty e^{-\nu t}(\nu t)^\nu \nu dt = \nu^{\nu+1}\int_0^\infty e^{\nu(-t+\ln t)}dt$. This last integral is of the form $\int_0^\infty \vartheta(t)e^{\nu h(t)}dt$ where ν is a large positive constant and $\vartheta(t)$ and $h(t)$ are real and continuous in $[0,\infty)$. Now $h(t) = -t + \ln t$ which has a single maximum at $t = 1$ with $h'(1) = 0$, $h''(1) = -1$. Applying the Laplace approximation to the two intervals $0 \leq t \leq 1$ and $1 \leq t < \infty$ we obtain the following:

$$\int_0^\infty e^{\nu(-t+\ln t)}dt = \int_0^1 e^{\nu(-t+\ln t)}dt + \int_1^\infty e^{\nu(-t+\ln t)}dt \,.$$

Since $h(t)$ has a maximum at $t = 1$, the main contributions to the two integrals on the right occur in the neighborhood of $t = 1$. We may rewrite $\int_0^1 e^{\nu(-t+\ln t)}dt = \int_0^{1-\varepsilon} e^{\nu(-t+\ln t)}dt + \int_{1-\varepsilon}^1 e^{\nu(-t+\ln t)}dt$ and $\int_1^\infty e^{\nu(-t+\ln t)}dt = \int_1^{1+\delta} e^{\nu(-t+\ln t)}dt + \int_{1+\delta}^\infty e^{\nu(-t+\ln t)}dt$. Setting $x^2 = h(1) - h(t)$, $2xdx = -h'(t)dt$ and expanding $h(t)$ in a Taylor's Series, $h(t) = h(1) + h'(1)(t-1) + h''(\xi)\,\frac{(t-1)^2}{2!}$, $1 < \xi < 1+\delta$, we obtain $x^2 = h(1) - h(t) = -h''(\xi)\,\frac{(t-1)^2}{2}$ and

$$h'(t) = h''(\xi)(t-1) \quad .$$

Now

$$\frac{2x}{h'(t)} = \frac{2\sqrt{-h''(\xi)\ \frac{(t-1)^2}{2}}}{h''(\xi)(t-1)}$$

$$= \frac{-1}{\sqrt{-\frac{1}{2}\ h''(\xi)}}$$

$$= \frac{-1}{\sqrt{-\frac{1}{2}\ h''(1)}} \quad .$$

Thus

$$\int_{t=1}^{1+\delta} e^{\nu(-t+\ell n\ t)} dt = \int_{x=0}^{\tau} e^{\nu(-1-x^2)}\left\{\frac{-2x}{h'(t)}\right\} dx$$

$$= e^{-\nu}\int_{\tau}^{0} e^{-\nu x^2} \frac{-1}{\sqrt{-\frac{1}{2}\ h''(1)}}\ dx$$

$$= e^{-\nu}\ \sqrt{2}\ \int_{0}^{\tau} e^{-\nu x^2}\ dx \ .$$

But $\int_0^\tau e^{-\nu x^2} dx \doteq \int_0^\infty e^{-\nu x^2}\ dx$ since the major contribution to this

latter integral occurs in the neighborhood of $x = 0$. Thus

$$\int_1^{1+\delta} e^{\nu(-t+\ln t)}dt \doteq e^{-\nu}\sqrt{2}\int_0^\infty e^{-\nu x^2}dx$$

$$= e^{-\nu}\sqrt{2}\left(\frac{1}{2}\sqrt{\frac{\pi}{\nu}}\right)$$

$$= e^{-\nu}\sqrt{\frac{\pi}{2\nu}} \quad .$$

Similarly we can show $\int_{1-\varepsilon}^{1} e^{\nu(-t+\ln t)}dt \doteq e^{-\nu}\sqrt{\frac{\pi}{2\nu}}$ and thus

$$\Gamma(\nu+1) \doteq \nu^{\nu+1}\left\{\int_{1-\varepsilon}^{1} e^{\nu(-t+\ln t)}\,dt + \int_1^{1+\delta} e^{\nu(-t+\ln t)}dt\right\}$$

$$= \nu^{\nu+1}\ 2e^{-\nu}\sqrt{\frac{\pi}{2\nu}}$$

$$= e^{-\nu}\nu^{\nu}\sqrt{2\pi\nu} \quad .$$

Appendix III: Evaluation of $\int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx$.

Theorem: $\int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx = 1/2$.

Proof: Let $f(v) = \int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx$.

$$\text{Then } f'(v) = \int_{-\infty}^{\infty} \phi(x)\phi(vx)xdx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-1/2(x^2+v^2x^2)}xdx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1+v^2}} \int_{-\infty}^{\infty} \frac{\sqrt{1+v^2}}{\sqrt{2\pi}} e^{-\frac{x^2}{2}(1+v^2)} xdx$$

and setting $u = x\sqrt{1+v^2}$ we obtain

$$f'(v) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1+v^2}} \int_{-\infty}^{\infty} z \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

$$= 0.$$

Then $f(v) = c$ for all v.

In particular, if $v = 1$,

$$f(1) = c = \int_{-\infty}^{\infty} \phi(x)\Phi(x)dx = \left.\frac{\Phi(x)^2}{2}\right|_{-\infty}^{\infty} = 1/2.$$

Thus $f(v) = 1/2$ for all v.

i.e. $\int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx = 1/2$.

Appendix IV

Tables of $u(r;n,k^*) = P(X_{(r)}$ is spurious in a sample of $n|k^*)$ where the exchangeable model of Weibulls with at most one outlier (involving shape change) is assumed.

$K^* = .01050$
$N = 50$

n

1 1.00000

2 .62989 .37011

3 .62721 .00535 .36743

4 .62566 .00470 .00334 .36632

5 .62454 .00444 .00273 .00263 .36666

6 .62368 .00431 .00249 .00205 .00226 .36521

7 .62297 .00422 .00237 .00182 .00171 .00202 .36487

8 .62238 .00417 .00229 .00171 .00148 .00151 .00186 .36461

9 .62186 .00412 .00223 .00164 .00137 .00127 .00138 .00173 .36439

10 .62141 .00409 .00219 .00159 .00131 .00120 .00110 .00128 .00163 .36421

11 .62100 .00407 .00216 .00154 .00127 .00110 .00100 .00110 .00120 .00164 .36406

12 .62063 .00405 .00213 .00151 .00123 .00110 .00095 .00091 .00099 .00110 .00146 .36392

13 .62030 .00403 .00211 .00148 .00120 .00100 .00092 .00084 .00084 .00095 .00110 .00140 .36381

14 .61999 .00402 .00210 .00146 .00120 .00100 .00091 .00081 .00075 .00079 .00092 .00100 .00134 .36370

15 .61970 .00401 .00209 .00145 .00110 .00098 .00089 .00080 .00072 .00069 .00076 .00089 .00099 .00129
.36361

16 .61943 .00400 .00208 .00144 .00110 .00095 .00087 .00079 .00071 .00064 .00065 .00075 .00086 .00095
.00126 .36353

17 .61918 .00399 .00207 .00143 .00110 .00093 .00084 .00076 .00071 .00063 .00059 .00063 .00073 .00084
.00090 .00120 .36345

18 .61895 .00398 .00206 .00142 .00110 .00091 .00081 .00076 .00070 .00063 .00057 .00056 .00061 .00072
.00081 .00086 .00120 .36338

19 .61873 .00397 .00205 .00142 .00110 .00089 .00079 .00074 .00070 .00063 .00056 .00052 .00053 .00061
.00071 .00078 .00082 .00120 .36332

20 .61852 .00396 .00204 .00141 .00110 .00088 .00077 .00071 .00068 .00063 .00057 .00051 .00048 .00051
.00060 .00070 .00075 .00078 .00110 .36326

21 .61832 .00396 .00204 .00140 .00110 .00087 .00075 .00069 .00066 .00063 .00068 .00061 .00046 .00046
.00051 .00060 .00069 .00072 .00075 .00110 .36320

22 .61814 .00395 .00203 .00140 .00110 .00087 .00074 .00067 .00064 .00062 .00058 .00062 .00046 .00043
.00044 .00050 .00060 .00067 .00069 .00072 .00110 .36315

23 .61796 .00395 .00202 .00139 .00110 .00087 .00073 .00065 .00062 .00061 .00058 .00053 .00047 .00042
.00040 .00043 .00060 .00060 .00066 .00066 .00069 .00110 .36310

24 .61778 .00394 .00202 .00139 .00110 .00087 .00072 .00064 .00080 .00059 .00067 .00063 .00048 .00042
.00039 .00038 .00042 .00051 .00060 .00064 .00063 .00067 .00110 .36306

25 .61762 .00394 .00201 .00138 .00110 .00087 .00072 .00062 .00058 .00067 .00066 .00054 .00049 .00043
.00039 .00036 .00037 .00042 .00051 .00059 .00062 .00061 .00066 .00110 .36301

26 .61746 .00394 .00201 .00138 .00110 .00087 .00072 .00062 .00056 .00055 .00064 .00053 .00049 .00044
.00039 .00036 .00034 .00037 .00043 .00051 .00058 .00060 .00068 .00063 .00100 .36297

27 .61731 .00394 .00201 .00137 .00110 .00087 .00073 .00061 .00055 .00053 .00063 .00052 .00050 .00046
.00040 .00036 .00033 .00033 .00036 .00043 .00052 .00058 .00068 .00056 .00061 .00100 .36293

28 .61717 .00393 .00200 .00136 .00110 .00087 .00073 .00061 .00054 .00051 .00051 .00061 .00050 .00046
.00041 .00036 .00033 .00031 .00032 .00036 .00044 .00052 .00056 .00055 .00063 .00060 .00100 .36289

29 .61703 .00393 .00200 .00136 .00110 .00087 .00073 .00062 .00063 .00049 .00049 .00048 .00049 .00047
.00042 .00037 .00033 .00031 .00030 .00032 .00037 .00044 .00062 .00055 .00053 .00051 .00058 .00100
.36266

30 .61689 .00393 .00200 .00136 .00100 .00087 .00073 .00062 .00053 .00048 .00047 .00047 .00048 .00047
.00043 .00039 .00034 .00031 .00029 .00029 .00031 .00037 .00046 .00062 .00064 .00061 .00049 .00067
.00100 .38282

31 .61676 .00393 .00200 .00136 .00100 .00087 .00074 .00062 .00053 .00047 .00046 .00046 .00047 .00046
.00044 .00040 .00035 .00031 .00029 .00028 .00028 .00031 .00038 .00046 .00052 .00062 .00049 .00048
.00056 .00100 .36279

32 .61663 .00392 .00199 .00135 .00100 .00086 .00074 .00063 .00053 .00047 .00044 .00044 .00046 .00045
.00044 .00041 .00036 .00032 .00029 .00027 .00027 .00026 .00032 .00038 .00046 .00061 .00060 .00047
.00046 .00056 .00100 .36275

33 .61661 .00392 .00199 .00134 .00100 .00086 .00074 .00063 .00064 .00046 .00043 .00042 .00043 .00044
.00044 .00041 .00037 .00033 .00029 .00027 .00026 .00026 .00027 .00032 .00039 .00047 .00050 .00048
.00045 .00046 .00056 .00100 .36272

34 .61639 .00392 .00199 .00134 .00100 .00085 .00074 .00064 .00054 .00046 .00042 .00040 .00041 .00043
.00043 .00042 .00038 .00034 .00030 .00027 .00026 .00025 .00025 .00027 .00033 .00040 .00047 .00049
.00047 .00043 .00043 .00064 .00100 .38269

35 .61628 .00392 .00199 .00134 .00100 .00086 .00074 .00064 .00054 .00047 .00041 .00039 .00039 .00041
.00042 .00042 .00039 .00035 .00031 .00028 .00026 .00024 .00024 .00026 .00028 .00033 .00041 .00047
.00048 .00046 .00041 .00042 .00063 .00100 .36266

36 .61616 .00392 .00199 .00134 .00100 .00084 .00073 .00064 .00055 .00047 .00041 .00038 .00038 .00039
.00041 .00041 .00040 .00036 .00032 .00028 .00026 .00024 .00023 .00023 .00024 .00028 .00034 .00042
.00047 .00047 .00043 .00039 .00041 .00053 .00100 .36263

37 .61606 .00391 .00199 .00134 .00100 .00084 .00073 .00064 .00055 .00047 .00041 .00037 .00036 .00037
.00039 .00040 .00040 .00037 .00033 .00029 .00026 .00024 .00023 .00022 .00022 .00024 .00028 .00035
.00042 .00046 .00045 .00041 .00038 .00041 .00052 .00100 .36260

38 .61595 .00391 .00199 .00134 .00100 .00083 .00073 .00064 .00066 .00048 .00041 .00037 .00035 .00036
.00038 .00039 .00040 .00038 .00035 .00030 .00027 .00025 .00023 .00022 .00022 .00022 .00024 .00029
.00036 .00043 .00046 .00044 .00039 .00037 .00040 .00052 .00100 .36258

39 .61585 .00391 .00198 .00134 .00100 .00083 .00072 .00084 .00056 .00048 .00041 .00037 .00034 .00034
.00036 .00038 .00039 .00038 .00035 .00032 .00028 .00025 .00023 .00022 .00021 .00021 .00022 .00024
.00030 .00037 .00043 .00045 .00042 .00037 .00035 .00039 .00061 .00100 .36256

40 .61575 .00391 .00198 .00133 .00100 .00082 .00072 .00064 .00057 .00049 .00042 .00037 .00034 .00033
.00034 .00036 .00038 .00038 .00038 .00033 .00029 .00026 .00023 .00022 .00021 .00021 .00020 .00021
.00025 .00031 .00038 .00043 .00044 .00040 .00036 .00034 .00039 .00061 .00100 .36252

41 .61565 .00391 .00198 .00133 .00100 .00082 .00071 .00064 .00057 .00049 .00042 .00037 .00033 .00032
.00033 .00035 .00037 .00037 .00037 .00034 .00030 .00026 .00024 .00022 .00021 .00020 .00020 .00020
.00021 .00025 .00032 .00038 .00043 .00043 .00039 .00034 .00033 .00038 .00061 .00100 .36250

42 .61555 .00391 .00198 .00133 .00100 .00081 .00071 .00064 .00057 .00050 .00043 .00037 .00033 .00031
.00031 .00033 .00035 .00037 .00037 .00035 .00031 .00027 .00024 .00022 .00021 .00020 .00020 .00019
.00019 .00022 .00026 .00033 .00039 .00043 .00042 .00037 .00033 .00033 .00038 .00051 .00100 .36248

43 .61546 .00391 .00198 .00133 .00100 .00081 .00070 .00063 .00057 .00050 .00043 .00037 .00033 .00031
.00030 .00031 .00033 .00035 .00036 .00035 .00032 .00029 .00025 .00022 .00021 .00020 .00020 .00019
.00019 .00019 .00022 .00027 .00034 .00040 .00042 .00040 .00036 .00031 .00032 .00037 .00050 .00100
.36245

44 .61537 .00390 .00198 .00133 .00100 .00081 .00070 .00063 .00057 .00051 .00044 .00038 .00033 .00030
.00029 .00030 .00032 .00034 .00035 .00035 .00033 .00030 .00026 .00023 .00021 .00020 .00020 .00019
.00018 .00018 .00019 .00022 .00028 .00034 .00040 .00042 .00039 .00034 .00030 .00031 .00037 .00050
.00100 .36243

45 .61528 .00390 .00198 .00133 .00100 .00080 .00069 .00063 .00057 .00051 .00045 .00038 .00033 .00030
.00029 .00029 .00030 .00032 .00034 .00035 .00034 .00031 .00027 .00024 .00021 .00020 .00019 .00019
.00018 .00018 .00018 .00019 .00023 .00029 .00035 .00040 .00041 .00037 .00032 .00029 .00031 .00037
.00050 .00100 .36240

46 .61519 .00390 .00198 .00133 .00100 .00080 .00069 .00062 .00057 .00051 .00045 .00039 .00034 .00030
.00028 .00028 .00029 .00031 .00033 .00034 .00034 .00032 .00029 .00025 .00022 .00020 .00019 .00019
.00018 .00018 .00017 .00017 .00019 .00024 .00030 .00036 .00040 .00040 .00036 .00031 .00028 .00030
.00036 .00050 .00100 .36238

47 .61511 .00390 .00198 .00133 .00100 .00080 .00068 .00062 .00057 .00062 .00046 .00039 .00034 .00030
.00028 .00027 .00028 .00029 .00032 .00033 .00034 .00033 .00030 .00026 .00023 .00020 .00019 .00018
.00018 .00018 .00017 .00017 .00017 .00020 .00024 .00031 .00037 .00040 .00039 .00034 .00029 .00027
.00030 .00036 .00049 .00100 .36236

48 .61503 .00390 .00198 .00133 .00100 .00080 .00068 .00061 .00057 .00052 .00046 .00040 .00034 .00030
.00027 .00026 .00026 .00028 .00030 .00032 .00033 .00033 .00031 .00027 .00024 .00021 .00019 .00018
.00018 .00018 .00017 .00017 .00016 .00017 .00020 .00025 .00032 .00037 .00040 .00038 .00033 .00028
.00026 .00030 .00036 .00049 .00100 .36234

49 .61496 .00390 .00197 .00133 .00100 .00080 .00067 .00061 .00056 .00052 .00046 .00040 .00035 .00030
.00027 .00026 .00026 .00027 .00029 .00031 .00033 .00033 .00031 .00029 .00025 .00022 .00019 .00018
.00018 .00018 .00017 .00017 .00016 .00016 .00017 .00021 .00026 .00033 .00038 .00039 .00037 .00031
.00027 .00026 .00030 .00036 .00049 .00100 .36232

60 .61487 .00390 .00197 .00133 .00100 .00080 .00067 .00060 .00056 .00062 .00047 .00041 .00036 .00031
.00027 .00025 .00025 .00025 .00027 .00030 .00032 .00033 .00032 .00030 .00026 .00023 .00020 .00018
.00017 .00017 .00017 .00017 .00016 .00015 .00015 .00017 .00021 .00027 .00033 .00038 .00038 .00035
.00030 .00025 .00025 .00029 .00035 .00049 .00100 .36230

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n

1 1.00000

2 .62979 .37021

3 .62698 .00561 .36741

4 .62534 .00492 .00349 .36625

5 .62418 .00466 .00286 .00275 .36556

6 .62327 .00451 .00261 .00215 .00236 .36509

7 .62254 .00443 .00248 .00191 .00180 .00212 .36473

8 .62191 .00436 .00240 .00179 .00156 .00158 .00194 .36446

9 .62137 .00432 .00234 .00172 .00144 .00134 .00144 .00181 .36423

10 .62090 .00429 .00229 .00166 .00137 .00120 .00120 .00133 .00170 .36404

11 .62047 .00426 .00226 .00162 .00132 .00120 .00110 .00110 .00125 .00161 .36388

12 .62008 .00424 .00223 .00158 .00128 .00110 .00100 .00096 .00100 .00120 .00154 .36374

13 .61973 .00422 .00221 .00155 .00125 .00110 .00097 .00088 .00089 .00099 .00110 .00147 .36362

14 .61941 .00421 .00220 .00153 .00120 .00110 .00095 .00085 .00080 .00084 .00095 .00110 .00141 .36351

15 .61911 .00420 .00219 .00152 .00120 .00100 .00092 .00084 .00076 .00074 .00081 .00092 .00100 .00137 .36341

16 .61883 .00419 .00218 .00151 .00120 .00099 .00090 .00082 .00074 .00068 .00070 .00078 .00089 .00097 .00133 .36332

17 .61857 .00418 .00216 .00150 .00120 .00097 .00087 .00081 .00074 .00067 .00063 .00067 .00077 .00086 .00093 .00129 .36324

18 .61832 .00417 .00216 .00149 .00120 .00095 .00085 .00079 .00073 .00066 .00060 .00059 .00065 .00075 .00083 .00088 .00126 .36317

19 .61809 .00416 .00215 .00148 .00110 .00094 .00082 .00077 .00072 .00066 .00059 .00055 .00056 .00064 .00074 .00080 .00085 .00124 .36310

20 .61787 .00415 .00214 .00148 .00110 .00093 .00080 .00074 .00071 .00066 .00060 .00054 .00051 .00055 .00063 .00072 .00076 .00081 .00120 .36303

21 .61767 .00415 .00213 .00147 .00110 .00092 .00079 .00072 .00069 .00065 .00060 .00054 .00049 .00049 .00054 .00063 .00071 .00073 .00078 .00120 .36297

22 .61747 .00414 .00213 .00146 .00110 .00092 .00078 .00070 .00066 .00064 .00060 .00055 .00049 .00046 .00047 .00054 .00063 .00069 .00070 .00075 .00120 .36292

23 .61728 .00414 .00212 .00146 .00110 .00092 .00077 .00068 .00064 .00063 .00060 .00055 .00049 .00045 .00043 .00046 .00054 .00062 .00067 .00067 .00072 .00120 .36287

24 .61710 .00413 .00211 .00145 .00110 .00092 .00077 .00067 .00062 .00061 .00059 .00056 .00050 .00045 .00041 .00041 .00046 .00054 .00061 .00065 .00064 .00070 .00120 .36281

25 .61693 .00413 .00211 .00144 .00110 .00092 .00077 .00066 .00060 .00058 .00058 .00056 .00051 .00046 .00041 .00039 .00040 .00046 .00054 .00061 .00063 .00062 .00068 .00110 .36277

26 .61677 .00413 .00211 .00144 .00110 .00092 .00077 .00066 .00059 .00056 .00056 .00055 .00052 .00047 .00042 .00038 .00037 .00039 .00046 .00054 .00060 .00060 .00059 .00066 .00110 .36272

27 .61661 .00412 .00210 .00143 .00110 .00092 .00077 .00065 .00058 .00054 .00054 .00054 .00052 .00048 .00043 .00038 .00035 .00035 .00039 .00046 .00054 .00058 .00058 .00057 .00065 .00110 .36268

28 .61645 .00412 .00210 .00143 .00110 .00091 .00077 .00065 .00057 .00053 .00052 .00052 .00051 .00048 .00044 .00039 .00035 .00034 .00035 .00039 .00047 .00054 .00057 .00056 .00054 .00063 .00110 .36264

29 .61631 .00412 .00210 .00142 .00110 .00091 .00077 .00066 .00057 .00052 .00050 .00050 .00050 .00048 .00045 .00040 .00036 .00033 .00032 .00034 .00039 .00047 .00054 .00055 .00053 .00052 .00062 .00110 .36260

30 .61617 .00411 .00209 .00142 .00110 .00091 .00077 .00066 .00057 .00051 .00048 .00048 .00049 .00048 .00045 .00041 .00036 .00033 .00031 .00031 .00034 .00040 .00048 .00053 .00054 .00051 .00051 .00061 .00110 .36256

31 .61603 .00411 .00209 .00142 .00110 .00090 .00077 .00066 .00057 .00050 .00047 .00046 .00047 .00047 .00046 .00042 .00037 .00033 .00031 .00029 .00030 .00034 .00041 .00048 .00052 .00052 .00049 .00049 .00060 .00110 .36252

32 .61590 .00411 .00209 .00141 .00110 .00090 .00077 .00067 .00057 .00050 .00046 .00045 .00046 .00046 .00046 .00043 .00038 .00034 .00031 .00029 .00028 .00030 .00034 .00041 .00048 .00051 .00050 .00047 .00048 .00060 .00110 .36248

33 .61577 .00411 .00209 .00141 .00110 .00089 .00077 .00067 .00057 .00050 .00045 .00043 .00044 .00045 .00045 .00043 .00040 .00035 .00031 .00029 .00027 .00028 .00030 .00035 .00042 .00048 .00050 .00048 .00045 .00046 .00059 .00110 .36245

34 .61564 .00411 .00209 .00141 .00110 .00089 .00077 .00067 .00058 .00050 .00044 .00042 .00042 .00043 .00044 .00043 .00040 .00036 .00032 .00029 .00027 .00026 .00027 .00030 .00035 .00043 .00048 .00049 .00046 .00043 .00045 .00058 .00110 .36241

35 .61552 .00410 .00209 .00141 .00110 .00088 .00076 .00067 .00058 .00050 .00044 .00041 .00040 .00041 .00043 .00043 .00041 .00037 .00033 .00030 .00027 .00026 .00025 .00027 .00030 .00036 .00043 .00048 .00048 .00044 .00042 .00044 .00058 .00110 .36238

36	.61540	.00410	.00208	.00141	.00110	.00088	.00076	.00067	.00059	.00051	.00044	.00040	.00039	.00040
	.00041	.00042	.00041	.00038	.00034	.00031	.00028	.00026	.00025	.00025	.00026	.00031	.00037	.00044
	.00047	.00046	.00042	.00040	.00043	.00057	.00110	.36235						
37	.61529	.00410	.00208	.00140	.00110	.00087	.00076	.00067	.00059	.00051	.00044	.00040	.00038	.00038
	.00040	.00041	.00041	.00039	.00036	.00032	.00028	.00026	.00024	.00024	.00024	.00026	.00031	.00038
	.00044	.00047	.00045	.00041	.00039	.00042	.00057	.00110	.36232					
38	.61518	.00410	.00208	.00140	.00110	.00087	.00075	.00067	.00059	.00052	.00045	.00039	.00037	.00037
	.00038	.00040	.00040	.00039	.00036	.00033	.00029	.00026	.00024	.00023	.00023	.00024	.00027	.00032
	.00039	.00044	.00046	.00043	.00039	.00038	.00042	.00057	.00110	.36229				
39	.61507	.00410	.00208	.00140	.00110	.00086	.00075	.00067	.00060	.00052	.00045	.00039	.00036	.00035
	.00036	.00038	.00039	.00039	.00037	.00034	.00030	.00027	.00025	.00023	.00023	.00022	.00024	.00027
	.00033	.00039	.00044	.00045	.00041	.00037	.00037	.00041	.00056	.00110	.36226			
40	.61497	.00410	.00208	.00140	.00110	.00086	.00074	.00066	.00060	.00053	.00045	.00040	.00036	.00034
	.00035	.00036	.00038	.00039	.00038	.00035	.00031	.00028	.00025	.00023	.00022	.00022	.00022	.00024
	.00027	.00033	.00040	.00044	.00043	.00040	.00036	.00036	.00041	.00056	.00110	.36223		
41	.61487	.00409	.00208	.00140	.00110	.00085	.00074	.00066	.00060	.00053	.00046	.00040	.00036	.00033
	.00033	.00035	.00037	.00038	.00038	.00036	.00032	.00029	.00025	.00023	.00022	.00021	.00021	.00021
	.00024	.00028	.00034	.00040	.00043	.00042	.00038	.00034	.00035	.00040	.00056	.00110	.36220	
42	.61477	.00409	.00208	.00140	.00110	.00085	.00073	.00066	.00060	.00053	.00047	.00040	.00036	.00033
	.00032	.00033	.00035	.00037	.00037	.00036	.00033	.00030	.00026	.00024	.00022	.00021	.00021	.00020
	.00021	.00024	.00029	.00035	.00041	.00043	.00041	.00036	.00033	.00034	.00040	.00056	.00110	.36218
43	.61467	.00409	.00207	.00140	.00110	.00085	.00072	.00065	.00060	.00054	.00047	.00041	.00036	.00032
	.00031	.00032	.00033	.00035	.00037	.00036	.00034	.00031	.00027	.00024	.00022	.00021	.00021	.00020
	.00020	.00021	.00024	.00030	.00036	.00041	.00042	.00039	.00035	.00032	.00034	.00039	.00056	.00110
	.36215													
44	.61457	.00409	.00207	.00140	.00110	.00085	.00072	.00065	.00059	.00054	.00048	.00041	.00036	.00032
	.00031	.00031	.00032	.00034	.00036	.00036	.00035	.00032	.00028	.00025	.00023	.00021	.00020	.00020
	.00019	.00020	.00021	.00025	.00031	.00037	.00041	.00041	.00038	.00033	.00031	.00033	.00039	.00056
	.00110	.36213												
45	.61448	.00409	.00207	.00140	.00110	.00084	.00072	.00064	.00059	.00054	.00048	.00042	.00036	.00032
	.00030	.00030	.00030	.00032	.00034	.00035	.00035	.00033	.00029	.00026	.00023	.00021	.00020	.00020
	.00019	.00019	.00019	.00021	.00026	.00031	.00037	.00041	.00040	.00036	.00032	.00030	.00033	.00039
	.00055	.00110	.36210											
46	.61439	.00409	.00207	.00140	.00110	.00084	.00071	.00064	.00059	.00054	.00048	.00042	.00037	.00032
	.00030	.00029	.00029	.00031	.00033	.00035	.00035	.00033	.00031	.00027	.00024	.00022	.00020	.00020
	.00019	.00019	.00018	.00019	.00022	.00026	.00032	.00038	.00040	.00039	.00035	.00030	.00029	.00032
	.00038	.00055	.00110	.36208										
47	.61430	.00409	.00207	.00140	.00110	.00084	.00071	.00063	.00058	.00054	.00049	.00043	.00037	.00033
	.00030	.00028	.00028	.00029	.00031	.00033	.00034	.00034	.00031	.00028	.00025	.00022	.00020	.00019
	.00019	.00019	.00018	.00018	.00019	.00022	.00027	.00033	.00038	.00040	.00038	.00033	.00029	.00029
	.00032	.00038	.00055	.00110	.36205									
48	.61421	.00408	.00207	.00140	.00110	.00084	.00070	.00063	.00058	.00054	.00049	.00043	.00038	.00033
	.00030	.00028	.00027	.00028	.00030	.00032	.00034	.00034	.00032	.00029	.00026	.00023	.00021	.00019
	.00019	.00018	.00018	.00017	.00018	.00019	.00023	.00028	.00034	.00038	.00039	.00036	.00032	.00028
	.00028	.00032	.00038	.00055	.00110	.36203								
49	.61413	.00408	.00207	.00140	.00110	.00084	.00070	.00062	.00057	.00054	.00049	.00044	.00038	.00033
	.00030	.00027	.00026	.00027	.00029	.00031	.00033	.00033	.00033	.00030	.00027	.00024	.00021	.00019
	.00019	.00018	.00018	.00017	.00017	.00017	.00019	.00023	.00029	.00035	.00038	.00038	.00035	.00030
	.00027	.00028	.00031	.00037	.00055	.00110	.36200							
50	.61404	.00408	.00207	.00140	.00110	.00084	.00070	.00062	.00057	.00053	.00049	.00044	.00039	.00034
	.00030	.00027	.00026	.00026	.00027	.00029	.00031	.00033	.00033	.00031	.00028	.00025	.00022	.00020
	.00019	.00018	.00018	.00017	.00017	.00017	.00017	.00020	.00024	.00030	.00035	.00038	.00038	.00034
	.00029	.00026	.00027	.00031	.00037	.00055	.00110	.36198						

$K^* = .01500$
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1 1.00000

2 .62894 .37106

3 .62511 .00765 .36724

4 .62288 .00671 .00476 .36565

5 .62129 .00635 .00390 .00375 .36471

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9 .61747 .00589 .00318 .00233 .00197 .00184 .00193 .00248 .36290

10 .61682 .00584 .00313 .00226 .00186 .00168 .00165 .00178 .00234 .36264

11 .61624 .00581 .00308 .00220 .00179 .00158 .00149 .00150 .00166 .00224 .36242

12 .61571 .00578 .00305 .00216 .00174 .00151 .00138 .00135 .00139 .00156 .00215 .36222

13 .61523 .00575 .00302 .00213 .00169 .00146 .00132 .00124 .00124 .00131 .00148 .00207 .36205

14 .61479 .00573 .00300 .00210 .00166 .00142 .00127 .00120 .00110 .00120 .00124 .00141 .00201 .36190

15 .61438 .00572 .00298 .00208 .00163 .00138 .00123 .00110 .00110 .00110 .00110 .00120 .00136 .00196 .36176

16 .61400 .00570 .00296 .00206 .00161 .00136 .00120 .00110 .00100 .00099 .00099 .00100 .00110 .00131 .00191 .36163

17 .61365 .00569 .00295 .00204 .00160 .00133 .00120 .00110 .00099 .00094 .00092 .00094 .00100 .00110 .00126 .00187 .36151

18 .61331 .00568 .00294 .00203 .00158 .00132 .00110 .00100 .00097 .00091 .00087 .00087 .00090 .00096 .00100 .00123 .00183 .36140

19 .61300 .00567 .00292 .00202 .00157 .00130 .00110 .00100 .00094 .00089 .00084 .00081 .00083 .00087 .00092 .00099 .00120 .00180 .36130

20 .61270 .00566 .00291 .00200 .00156 .00129 .00110 .00099 .00092 .00087 .00082 .00078 .00077 .00079 .00084 .00088 .00095 .00120 .00177 .36121

21 .61242 .00565 .00291 .00199 .00154 .00128 .00110 .00098 .00090 .00085 .00080 .00076 .00073 .00073 .00077 .00082 .00085 .00092 .00110 .00174 .36112

22 .61215 .00564 .00290 .00199 .00153 .00127 .00110 .00096 .00088 .00082 .00079 .00075 .00071 .00069 .00070 .00075 .00079 .00082 .00089 .00110 .00171 .36104

23 .61189 .00564 .00289 .00198 .00153 .00126 .00110 .00095 .00086 .00081 .00077 .00073 .00070 .00066 .00065 .00068 .00073 .00077 .00079 .00086 .00110 .00169 .36097

24 .61165 .00563 .00289 .00197 .00152 .00125 .00110 .00095 .00085 .00079 .00075 .00072 .00069 .00065 .00062 .00063 .00067 .00071 .00074 .00076 .00084 .00110 .00167 .36089

25 .61141 .00563 .00288 .00197 .00151 .00124 .00110 .00094 .00084 .00077 .00073 .00070 .00068 .00064 .00061 .00059 .00061 .00065 .00070 .00072 .00073 .00082 .00110 .00165 .36082

26 .61119 .00562 .00287 .00196 .00150 .00124 .00110 .00093 .00084 .00076 .00071 .00069 .00067 .00064 .00060 .00057 .00057 .00060 .00064 .00068 .00069 .00071 .00080 .00110 .00163 .36076

27 .61097 .00562 .00287 .00195 .00150 .00123 .00110 .00093 .00083 .00075 .00070 .00067 .00065 .00063 .00060 .00056 .00054 .00055 .00059 .00063 .00066 .00067 .00068 .00079 .00110 .00161 .36070

28 .61076 .00561 .00286 .00195 .00149 .00122 .00100 .00092 .00083 .00075 .00069 .00065 .00064 .00062 .00060 .00056 .00053 .00052 .00054 .00058 .00062 .00064 .00064 .00066 .00078 .00110 .00159 .36064

29 .61056 .00561 .00286 .00195 .00149 .00120 .00100 .00092 .00082 .00074 .00068 .00064 .00062 .00061 .00059 .00056 .00053 .00050 .00050 .00053 .00057 .00061 .00062 .00062 .00064 .00077 .00100 .00157 .36058

30 .61037 .00561 .00286 .00194 .00149 .00120 .00100 .00091 .00082 .00074 .00067 .00063 .00060 .00059 .00058 .00056 .00053 .00050 .00048 .00049 .00052 .00057 .00060 .00060 .00060 .00063 .00076 .00100 .00155 .36053

31 .61018 .00560 .00285 .00194 .00148 .00120 .00100 .00091 .00081 .00074 .00067 .00062 .00059 .00058 .00057 .00055 .00053 .00049 .00047 .00046 .00048 .00052 .00056 .00059 .00058 .00058 .00061 .00076 .00100 .00153 .36048

32 .61000 .00560 .00285 .00193 .00148 .00120 .00100 .00090 .00081 .00074 .00067 .00061 .00058 .00056 .00055 .00055 .00053 .00050 .00047 .00044 .00045 .00047 .00052 .00055 .00057 .00056 .00056 .00060 .00075 .00100 .00151 .36043

33 .60983 .00560 .00285 .00193 .00147 .00120 .00100 .00089 .00081 .00073 .00067 .00061 .00057 .00055 .00054 .00054 .00052 .00050 .00047 .00044 .00043 .00044 .00047 .00051 .00055 .00056 .00054 .00054 .00059 .00075 .00100 .00150 .36038

34 .60966 .00559 .00284 .00193 .00147 .00120 .00100 .00089 .00080 .00073 .00067 .00061 .00056 .00053 .00052 .00052 .00052 .00050 .00047 .00044 .00042 .00041 .00043 .00047 .00051 .00054 .00054 .00053 .00052 .00058 .00074 .00100 .00148 .36034

35 .60949 .00559 .00284 .00193 .00147 .00120 .00100 .00088 .00079 .00073 .00067 .00061 .00056 .00053 .00051 .00051 .00051 .00050 .00048 .00044 .00041 .00040 .00040 .00043 .00047 .00051 .00053 .00052 .00051 .00051 .00057 .00074 .00100 .00147 .36029

36 .60934 .00559 .00284 .00192 .00147 .00120 .00100 .00088 .00079 .00072 .00066 .00061 .00056 .00052
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37 .60918 .00558 .00284 .00192 .00146 .00120 .00100 .00088 .00078 .00072 .00066 .00061 .00056 .00051
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38 .60903 .00558 .00283 .00192 .00146 .00120 .00100 .00087 .00078 .00071 .00066 .00061 .00056 .00051
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39 .60888 .00558 .00283 .00192 .00146 .00120 .00100 .00087 .00077 .00071 .00066 .00061 .00056 .00051
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40 .60874 .00558 .00283 .00191 .00146 .00120 .00100 .00087 .00077 .00070 .00065 .00061 .00056 .00051
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41 .60860 .00558 .00283 .00191 .00145 .00120 .00100 .00086 .00077 .00070 .00065 .00061 .00056 .00051
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42 .60846 .00557 .00283 .00191 .00145 .00120 .00100 .00086 .00076 .00069 .00064 .00060 .00056 .00052
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.00094 .00132 .35994

46 .60795 .00557 .00282 .00190 .00144 .00120 .00099 .00086 .00076 .00068 .00062 .00059 .00056 .00052
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.00074 .00093 .00131 .35991

47 .60783 .00557 .00282 .00190 .00144 .00120 .00099 .00086 .00075 .00068 .00062 .00058 .00055 .00052
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.00028 .00027 .00028 .00030 .00033 .00037 .00040 .00042 .00041 .00039 .00037 .00036 .00037 .00043
.00056 .00074 .00092 .00130 .35988

48 .60771 .00556 .00282 .00190 .00144 .00120 .00099 .00086 .00075 .00067 .00062 .00058 .00055 .00052
.00049 .00045 .00041 .00038 .00036 .00036 .00036 .00037 .00038 .00038 .00037 .00035 .00033 .00030
.00028 .00027 .00027 .00028 .00030 .00034 .00037 .00040 .00041 .00040 .00038 .00035 .00035 .00037
.00043 .00056 .00074 .00091 .00129 .35986

49 .60760 .00556 .00281 .00190 .00144 .00120 .00099 .00086 .00075 .00067 .00061 .00057 .00054 .00052
.00049 .00045 .00042 .00038 .00036 .00035 .00035 .00036 .00037 .00038 .00037 .00036 .00034 .00031
.00029 .00027 .00026 .00026 .00028 .00031 .00034 .00038 .00040 .00040 .00039 .00036 .00034 .00034
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50 .60748 .00556 .00281 .00190 .00144 .00120 .00098 .00086 .00075 .00067 .00061 .00057 .00054 .00052
.00049 .00046 .00042 .00038 .00036 .00034 .00034 .00035 .00036 .00037 .00037 .00036 .00034 .00032
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n

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3 .62278 .01020 .36703

4 .61980 .00894 .00635 .36491

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9 .61259 .00785 .00425 .00311 .00263 .00246 .00257 .00331 .36124

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11 .61095 .00774 .00411 .00294 .00239 .00211 .00198 .00200 .00221 .00298 .36060

12 .61025 .00770 .00406 .00288 .00231 .00201 .00185 .00179 .00185 .00209 .00286 .36034

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15 .60848 .00762 .00397 .00277 .00218 .00184 .00164 .00151 .00144 .00141 .00145 .00156 .00183 .00260
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16 .60797 .00760 .00395 .00274 .00215 .00181 .00159 .00146 .00137 .00133 .00133 .00137 .00149 .00177
.00253 .35955

17 .60750 .00758 .00393 .00272 .00213 .00178 .00156 .00142 .00132 .00126 .00124 .00125 .00131 .00143
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18 .60706 .00756 .00391 .00270 .00211 .00176 .00153 .00138 .00128 .00120 .00120 .00120 .00120 .00125
.00138 .00166 .00242 .35925

19 .60664 .00755 .00390 .00269 .00209 .00174 .00151 .00135 .00125 .00120 .00110 .00110 .00110 .00110
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20 .60624 .00754 .00388 .00267 .00207 .00172 .00149 .00133 .00120 .00110 .00110 .00110 .00100 .00110
.00110 .00120 .00130 .00158 .00233 .35900

21 .60586 .00753 .00387 .00266 .00206 .00170 .00147 .00131 .00120 .00110 .00110 .00100 .00099 .00099
.00100 .00100 .00110 .00126 .00155 .00229 .35888

22 .60551 .00752 .00386 .00265 .00204 .00169 .00146 .00129 .00120 .00110 .00100 .00099 .00096 .00094
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23 .60516 .00751 .00385 .00264 .00203 .00168 .00144 .00128 .00120 .00110 .00100 .00096 .00093 .00091
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24 .60484 .00750 .00384 .00263 .00202 .00166 .00143 .00127 .00110 .00110 .00099 .00094 .00091 .00088
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25 .60453 .00749 .00384 .00262 .00201 .00166 .00142 .00125 .00110 .00100 .00097 .00092 .00088 .00085
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26 .60423 .00749 .00383 .00261 .00201 .00165 .00141 .00124 .00110 .00100 .00096 .00090 .00087 .00084
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27 .60394 .00748 .00382 .00261 .00200 .00164 .00140 .00123 .00110 .00100 .00094 .00089 .00085 .00082
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28 .60366 .00747 .00382 .00260 .00199 .00163 .00139 .00122 .00110 .00100 .00093 .00087 .00083 .00080
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29 .60340 .00747 .00381 .00259 .00199 .00162 .00138 .00120 .00110 .00100 .00092 .00086 .00082 .00079
.00076 .00074 .00072 .00070 .00071 .00072 .00074 .00076 .00078 .00083 .00093 .00110 .00134 .00206
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30 .60314 .00746 .00381 .00259 .00198 .00162 .00138 .00120 .00110 .00099 .00091 .00085 .00081 .00077
.00075 .00072 .00070 .00069 .00068 .00068 .00070 .00072 .00074 .00076 .00081 .00092 .00110 .00132
.00203 .35810

31 .60289 .00746 .00380 .00258 .00197 .00161 .00137 .00120 .00110 .00098 .00091 .00085 .00080 .00076
.00073 .00071 .00069 .00067 .00066 .00065 .00067 .00069 .00070 .00072 .00074 .00080 .00091 .00110
.00130 .00201 .35803

32 .60265 .00745 .00380 .00258 .00197 .00161 .00137 .00120 .00110 .00097 .00090 .00084 .00079 .00075
.00072 .00070 .00068 .00066 .00064 .00063 .00064 .00065 .00067 .00069 .00070 .00072 .00079 .00090
.00100 .00129 .00200 .35797

33 .60242 .00745 .00379 .00257 .00197 .00160 .00136 .00120 .00110 .00097 .00089 .00083 .00078 .00074
.00071 .00068 .00067 .00065 .00063 .00062 .00061 .00062 .00064 .00066 .00067 .00068 .00070 .00077
.00089 .00100 .00127 .00198 .35791

34 .60219 .00745 .00379 .00257 .00196 .00160 .00136 .00120 .00110 .00096 .00089 .00083 .00077 .00073
.00070 .00067 .00066 .00064 .00063 .00061 .00059 .00059 .00061 .00063 .00064 .00065 .00066 .00069
.00077 .00088 .00100 .00125 .00196 .35785

35 .60197 .00744 .00378 .00257 .00196 .00159 .00135 .00120 .00110 .00096 .00088 .00082 .00077 .00072
.00069 .00066 .00064 .00063 .00062 .00060 .00058 .00057 .00058 .00059 .00061 .00063 .00063 .00064
.00068 .00076 .00087 .00100 .00124 .00195 .35779

36	.60176	.00744	.00378	.00256	.00195	.00159	.00135	.00120	.00100	.00095	.00087	.00081	.00076	.00072
	.00068	.00065	.00063	.00062	.00061	.00059	.00057	.00056	.00056	.00056	00058	.00060	.00061	.00062
	.00063	.00067	.00075	.00086	.00098	.00122	.00193	.35774						
37	.60155	.00743	.00378	.00256	.00195	.00159	.00134	.00120	.00100	.00095	.00087	.00081	.00076	.00071
	.00068	.00064	.00062	.00061	.00060	.00059	.00057	.00055	.00054	.00054	.00055	.00057	.00059	.00060
	.00060	.00061	.00066	.00074	.00085	.00097	.00120	.00192	.35768					
38	.60135	.00743	.00377	.00255	.00195	.00158	.00134	.00120	.00100	.00094	.00086	.00080	.00075	.00071
	.00067	.00064	.00061	.00060	.00059	.00058	.00056	.00055	.00053	.00052	.00053	.00055	.00057	.00058
	.00058	.00058	.00060	.00065	.00074	.00084	.00096	.00120	.00190	.35763				
39	.60116	.00743	.00377	.00255	.00194	.00158	.00134	.00120	.00100	.00094	.00086	.00080	.00075	.00071
	.00067	.00063	.00060	.00058	.00057	.00057	.00056	.00054	.00052	.00051	.00051	.00052	.00054	.00056
	.00057	.00057	.00057	.00059	.00064	.00073	.00083	.00094	.00120	.00189	.35758			
40	.60097	.00743	.00377	.00255	.00194	.00158	.00133	.00120	.00100	.00093	.00085	.00079	.00074	.00070
	.00066	.00063	.00060	.00058	.00056	.00056	.00055	.00054	.00052	.00050	.00049	.00050	.00051	.00053
	.00055	.00056	.00055	.00055	.00058	.00064	.00073	.00082	.00093	.00120	.00188	.35753		
41	.60078	.00742	.00377	.00255	.00194	.00157	.00133	.00120	.00100	.00093	.00085	.00079	.00074	.00070
	.00066	.00063	.00059	.00057	.00055	.00054	.00054	.00053	.00052	.00050	.00049	.00048	.00049	.00050
	.00053	.00054	.00054	.00054	.00054	.00057	.00063	.00072	.00081	.00092	.00120	.00187	.35749	
42	.60060	.00742	.00376	.00254	.00193	.00157	.00133	.00120	.00100	.00093	.00085	.00078	00073	.00069
	.00066	.00062	.00059	.00056	.00054	.00053	.00053	.00052	.00051	.00050	.00048	.00047	.00047	.00048
	.00050	.00052	.00053	.00053	.00052	.00053	.00056	.00063	.00072	.00080	.00091	.00120	.00186	.35744
43	.60042	.00742	.00376	.00254	.00193	.00157	.00132	.00120	.00100	.00092	.00084	.00078	.00073	.00069
	.00065	.00062	.00059	.00056	.00054	.00052	.00052	.00052	.00051	.00050	.00048	.00046	.00045	.00046
	.00047	.00049	.00051	.00052	.00052	.00051	.00052	.00056	.00063	.00071	.00079	00090	.00110	.00185
	.35740													
44	.60025	.00742	.00376	.00254	.00193	.00156	.00132	.00110	.00100	.00092	00084	.00078	.00072	.00068
	.00065	.00062	.00059	.00056	.00053	.00052	.00051	.00051	.00050	.00049	.00048	.00046	.00045	.00044
	.00045	.00047	.00049	.00051	.00051	.00050	.00050	.00051	.00055	.00062	.00071	.00079	.00088	.00110
	.00184	.35736												
45	.60008	.00741	.00376	.00254	.00193	.00156	.00132	.00110	.00100	.00092	.00084	.00077	.00072	.00068
	.00064	.00061	.00058	.00056	.00053	.00051	.00050	.00049	.00049	.00049	.00048	.00046	.00044	.00043
	.00043	.00044	.00046	.00049	.00050	.00050	.00049	.00049	.00050	.00055	.00062	.00070	.00078	.00087
	.00110	.00183	.35731											
46	.59992	.00741	.00375	.00253	.00192	.00156	.00132	.00110	.00100	.00092	.00084	.00077	.00072	.00067
	.00064	.00061	.00058	.00055	.00053	.00050	.00049	.00048	.00048	.00048	.00047	.00046	.00044	.00043
	.00042	.00042	.00044	.00046	.00048	.00049	.00049	.00048	.00048	.00049	.00054	.00062	.00070	00077
	.00086	.00110	.00182	.35727										
47	.59976	.00741	.00375	.00253	.00192	.00156	.00131	.00110	00100	.00091	.00083	.00077	.00071	.00067
	.00063	.00061	.00058	.00055	.00053	.00050	.00048	.00048	.00047	.00047	.00047	.00046	.00044	.00043
	.00041	.00041	.00042	.00044	.00046	.00048	.00048	.00047	.00047	.00046	.00049	.00054	.00062	.00069
	.00076	.00085	.00110	.00181	.35723									
48	.59960	.00741	.00375	.00253	.00192	.00156	.00131	.00110	.00100	.00091	.00083	.00077	.00071	.00066
	.00063	.00060	.00058	.00055	.00052	.00050	.00048	.00047	.00046	.00046	.00046	.00046	.00044	.00043
	.00041	.00040	.00040	.00041	.00043	.00046	.00047	.00047	.00046	.00045	.00046	.00048	.00054	.00061
	.00069	.00075	.00084	.00110	.00180	.35720								
49	.59944	.00740	.00375	.00253	.00192	.00155	.00131	.00110	.00100	.00091	.00083	.00076	.00071	.00066
	.00063	.00060	.00057	.00055	.00052	.00050	.00048	.00046	.00045	.00045	.00045	00045	.00044	.00043
	.00041	.00040	.00039	.00040	.00041	.00043	.00045	.00046	.00046	.00045	.00044	00045	.00048	.00054
	.00061	.00068	.00074	.00083	.00110	.00179	.35716							
50	.59929	.00740	.00375	.00253	.00192	.00155	.00131	.00110	.00100	.00091	.00083	.00076	00071	.00066
	.00062	.00059	.00057	.00055	.00052	.00050	.00047	.00046	.00045	.00044	.00045	.00045	.00044	.00043
	.00041	.00040	.00038	.00038	.00039	.00041	.00043	.00045	.00046	.00045	00044	.00043	00044	.00048
	.00054	.00061	.00068	.00073	00082	.00110	.00179	.35712						

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1 1.00000

2 .62154 .37846

3 .60883 .02542 .36575

4 .60141 .02227 .01586 .36046

5 .59615 .02104 .01298 .01249 .35734

6 .59207 .02038 .01185 .00979 .01070 .35519

7 .58874 .01996 .01120 .00872 .00814 .00961 .35359

8 .58593 .01967 .01080 .00814 .00712 .00713 .00883 .35233

9 .58350 .01946 .01060 .00777 .00656 .00614 .00643 .00826 .35130

10 .58136 .01929 .01040 .00751 .00621 .00560 .00548 .00592 .00781 .35043

11 .57944 .01916 .01020 .00732 .00596 .00526 .00496 .00500 .00553 .00745 .34968

12 .57771 .01906 .01010 .00718 .00578 .00502 .00462 .00449 .00463 .00522 .00715 .34903

13 .57613 .01897 .01000 .00707 .00564 .00485 .00439 .00416 .00413 .00433 .00496 .00690 .34846

14 .57467 .01889 .00994 .00697 .00553 .00471 .00422 .00393 .00381 .00385 .00409 .00475 .00668 .34794

15 .57333 .01883 .00987 .00690 .00544 .00460 .00409 .00377 .00359 .00353 .00362 .00390 .00457 .00649 .34748

16 .57208 .01877 .00981 .00683 .00537 .00452 .00398 .00364 .00343 .00332 .00331 .00343 .00373 .00441 .00632 .34706

17 .57091 .01872 .00976 .00678 .00530 .00444 .00390 .00354 .00330 .00316 .00310 .00313 .00327 .00358 .00427 .00618 .34667

18 .56981 .01867 .00972 .00673 .00525 .00438 .00383 .00345 .00320 .00303 .00294 .00292 .00297 .00313 .00346 .00415 .00604 .34632

19 .56877 .01863 .00968 .00669 .00520 .00433 .00377 .00338 .00312 .00293 .00282 .00276 .00276 .00284 .00301 .00335 .00404 .00592 .34599

20 .56780 .01860 .00965 .00665 .00516 .00429 .00372 .00332 .00305 .00285 .00272 .00264 .00261 .00263 .00272 .00290 .00325 .00394 .00582 .34568

21 .56687 .01856 .00961 .00662 .00513 .00425 .00367 .00327 .00299 .00279 .00264 .00255 .00249 .00248 .00252 .00262 .00281 .00316 .00385 .00572 .34540

22 .56599 .01853 .00959 .00659 .00510 .00421 .00363 .00323 .00294 .00273 .00258 .00247 .00240 .00237 .00237 .00242 .00253 .00273 .00308 .00377 .00562 .34513

23 .56514 .01850 .00956 .00656 .00507 .00418 .00360 .00319 .00290 .00268 .00252 .00240 .00232 .00227 .00226 .00228 .00234 .00245 .00265 .00301 .00370 .00554 .34488

24 .56434 .01848 .00954 .00654 .00504 .00415 .00357 .00316 .00286 .00264 .00247 .00235 .00226 .00220 .00217 .00216 .00219 .00226 .00238 .00258 .00294 .00363 .00546 .34464

25 .56357 .01845 .00952 .00652 .00502 .00413 .00354 .00313 .00283 .00260 .00243 .00230 .00220 .00213 .00209 .00207 .00208 .00211 .00219 .00231 .00252 .00288 .00357 .00539 .34442

26 .56283 .01843 .00950 .00650 .00500 .00410 .00351 .00310 .00280 .00257 .00239 .00226 .00216 .00208 .00203 .00200 .00199 .00200 .00205 .00213 .00226 .00247 .00282 .00351 .00532 .34420

27 .56213 .01841 .00948 .00648 .00498 .00408 .00349 .00308 .00277 .00254 .00236 .00222 .00212 .00203 .00197 .00194 .00192 .00191 .00194 .00199 .00207 .00220 .00241 .00277 .00346 .00526 .34400

28 .56145 .01839 .00946 .00646 .00496 .00406 .00347 .00305 .00275 .00251 .00233 .00219 .00208 .00199 .00193 .00188 .00185 .00184 .00185 .00188 .00193 .00202 .00215 .00237 .00272 .00341 .00520 .34381

29 .56079 .01837 .00944 .00644 .00494 .00405 .00345 .00303 .00273 .00249 .00231 .00216 .00205 .00196 .00189 .00184 .00180 .00178 .00178 .00179 .00182 .00188 .00197 .00211 .00232 .00268 .00336 .00514 .34362

30 .56016 .01835 .00943 .00643 .00493 .00403 .00344 .00302 .00271 .00247 .00228 .00214 .00202 .00193 .00185 .00180 .00176 .00173 .00172 .00172 .00174 .00177 .00183 .00192 .00206 .00228 .00264 .00332 .00509 .34345

31 .55955 .01834 .00942 .00642 .00491 .00402 .00342 .00300 .00269 .00245 .00226 .00211 .00200 .00190 .00182 .00176 .00172 .00169 .00167 .00166 .00167 .00169 .00173 .00179 .00188 .00203 .00224 .00260 .00327 .00504 .34328

32 .55895 .01832 .00940 .00640 .00490 .00400 .00341 .00298 .00267 .00243 .00224 .00209 .00197 .00188 .00180 .00173 .00168 .00165 .00162 .00161 .00161 .00162 .00164 .00168 .00175 .00185 .00199 .00221 .00256 .00324 .00499 .34312

33 .55838 .01831 .00939 .00639 .00489 .00399 .00339 .00297 .00266 .00241 .00223 .00207 .00195 .00185 .00177 .00171 .00165 .00161 .00158 .00156 .00156 .00156 .00157 .00160 .00165 .00171 .00181 .00195 .00217 .00253 .00320 .00494 .34297

34 .55783 .01829 .00938 .00638 .00488 .00398 .00338 .00296 .00264 .00240 .00221 .00206 .00193 .00183 .00175 .00168 .00163 .00158 .00155 .00153 .00151 .00151 .00151 .00153 .00156 .00161 .00168 .00178 .00192 .00214 .00250 .00316 .00490 .34282

35 .55729 .01828 .00937 .00637 .00487 .00397 .00337 .00294 .00263 .00239 .00219 .00204 .00192 .00181 .00173 .00166 .00160 .00156 .00152 .00149 .00148 .00147 .00147 .00147 .00149 .00153 .00158 .00165 .00175 .00189 .00211 .00246 .00313 .00486 .34267

36	.55677	.01827	.00936	.00636	.00486	.00396	.00336	.00293	.00262	.00237	.00218	.00203	.00190	.00180
	.00171	.00164	.00158	.00153	.00149	.00146	.00144	.00143	.00142	.00143	.00144	.00146	.00149	.00155
	.00162	.00172	.00186	.00208	.00244	.00310	.00482	.34254						
37	.55626	.01826	.00935	.00635	.00485	.00395	.00335	.00292	.00261	.00236	.00217	.00201	.00189	.00178
	.00169	.00162	.00156	.00151	.00147	.00144	.00141	.00140	.00139	.00138	.00139	.00140	.00143	.00146
	.00152	.00159	.00169	.00184	.00206	.00241	.00307	.00478	.34240					
38	.55577	.01824	.00934	.00634	.00484	.00394	.00334	.00291	.00259	.00235	.00216	.00200	.00187	.00177
	.00168	.00160	.00154	.00149	.00145	.00141	.00139	.00137	.00135	.00135	.00135	.00135	.00137	.00140
	.00144	.00149	.00156	.00167	.00181	.00203	.00238	.00304	.00475	.34228				
39	.55529	.01823	.00933	.00633	.00483	.00393	.00333	.00290	.00258	.00234	.00215	.00199	.00186	.00175
	.00166	.00159	.00153	.00147	.00143	.00139	.00136	.00134	.00132	.00131	.00131	.00131	.00132	.00134
	.00137	.00141	.00146	.00154	.00164	.00179	.00201	.00236	.00301	.00471	.34215			
40	.55482	.01822	.00932	.00632	.00482	.00392	.00332	.00289	.00258	.00233	.00213	.00198	.00185	.00174
	.00165	.00158	.00151	.00146	.00141	.00137	.00134	.00132	.00130	.00129	.00128	.00128	.00128	.00129
	.00131	.00134	.00138	.00144	.00151	.00162	.00177	.00198	.00233	.00298	.00468	.34203		
41	.55437	.01821	.00931	.00632	.00481	.00391	.00331	.00289	.00257	.00232	.00213	.00197	.00184	.00173
	.00164	.00156	.00150	.00144	.00140	.00136	.00132	.00130	.00128	.00126	.00125	.00125	.00125	.00125
	.00127	.00129	.00132	.00136	.00142	.00149	.00160	.00174	.00196	.00231	.00296	.00465	.34192	
42	.55392	.01820	.00931	.00631	.00481	.00390	.00330	.00288	.00256	.00231	.00212	.00196	.00183	.00172
	.00163	.00155	.00148	.00143	.00138	.00134	.00131	.00128	.00126	.00124	.00123	.00120	.00120	.00120
	.00123	.00124	.00126	.00130	.00134	.00140	.00147	.00158	.00172	.00194	.00229	.00294	.00462	.34180
43	.55349	.01819	.00930	.00630	.00480	.00390	.00330	.00287	.00255	.00230	.00211	.00195	.00182	.00171
	.00162	.00154	.00147	.00142	.00137	.00133	.00129	.00126	.00124	.00120	.00120	.00120	.00120	.00120
	.00120	.00120	.00120	.00124	.00127	.00132	.00137	.00145	.00156	.00170	.00192	.00227	.00291	.00459
	.34169													
44	.55307	.01818	.00929	.00630	.00479	.00389	.00329	.00286	.00254	.00230	.00210	.00194	.00181	.00170
	.00161	.00153	.00146	.00140	.00135	.00131	.00128	.00124	.00120	.00120	00120	.00120	.00120	.00120
	.00120	.00120	.00120	.00120	.00120	.00125	.00130	.00136	.00143	.00154	.00168	.00190	.00225	.00289
	.00456	.34159												
45	.55265	.01817	.00928	.00629	.00479	.00388	.00328	.00286	.00254	.00229	.00209	.00193	.00180	.00169
	.00160	.00152	.00145	.00139	.00134	.00130	.00126	.00123	.00120	.00120	.00120	.00110	.00110	.00110
	.00110	.00110	.00110	.00120	.00120	.00120	.00123	.00128	.00134	.00142	.00152	.00167	.00188	.00223
	.00287	.00453	.34149											
46	55225	.01816	.00928	.00628	.00478	.00388	.00328	.00285	.00253	.00228	00208	.00192	.00179	.00168
	.00159	.00151	.00144	.00138	.00133	.00129	.00125	.00120	.00120	.00120	.00110	.00110	.00110	.00110
	.00110	.00110	.00110	.00110	.00110	.00120	.00120	.00120	.00126	.00132	.00140	.00150	.00165	.00187
	.00221	.00285	.00450	.34139										
47	.55185	.01816	.00927	.00628	.00478	.00387	.00327	.00284	.00252	.00228	.00208	.00192	.00179	.00167
	.00158	.00150	.00143	.00137	.00132	.00128	.00124	.00120	.00120	.00120	.00110	00110	.00110	.00110
	.00110	.00110	.00110	.00110	.00110	.00110	.00110	.00120	.00120	.00124	.00130	.00138	.00149	.00163
	.00185	.00219	.00283	.00448	.34129									
48	55147	.01815	.00927	.00627	.00477	.00387	.00327	.00284	.00252	.00227	.00207	.00191	.00178	.00167
	.00157	.00149	.00142	.00136	.00131	.00127	.00123	.00120	.00120	.00110	.00110	.00110	.00110	.00110
	.00110	.00110	.00110	.00110	.00110	.00110	.00110	.00110	.00110	.00120	.00123	.00129	.00137	.00147
	.00162	.00183	.00217	.00281	.00445	.34119								
49	.55109	.01814	.00926	.00627	.00476	.00386	.00326	.00283	.00251	.00226	.00207	.00190	.00177	.00166
	.00157	.00148	.00141	.00135	.00130	.00126	.00120	.00120	.00120	.00110	.00110	.00110	.00110	.00110
	.00100	.00100	.00100	.00100	.00100	.00110	.00110	.00110	.00110	.00110	.00120	.00120	.00127	.00135
	.00146	.00160	.00182	.00216	.00279	.00443	.34110							
50	.55072	.01813	.00925	.00626	.00476	.00386	.00326	.00283	.00251	00226	.00206	.00190	.00176	.00165
	.00156	00148	.00141	.00135	.00129	.00125	.00120	.00120	.00110	.00110	.00110	.00110	.00110	.00100
	.00100	.00100	.00100	.00100	.00100	.00100	.00100	.00100	.00110	.00110	.00110	.00110	.00120	.00126
	.00134	.00144	.00159	.00180	.00214	.00277	.00441	.34101						

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2 .61118 .38882

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6 .55295 .03999 .02357 .01954 .02140 .34254

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8 .54095 .03840 .02151 .01621 .01422 .01424 .01763 .33683

9 .53621 .03790 .02095 .01546 .01310 .01228 .01285 .01647 .33477

10 .53205 .03750 .02054 .01494 .01238 .01120 .01100 .01183 .01557 .33304

11 .52833 .03718 .02021 .01456 .01188 .01050 .00990 .00998 .01100 .01485 .33156

12 .52497 .03691 .01995 .01426 .01150 .01000 .00923 .00897 .00924 .01040 .01425 .33026

13 .52192 .03667 .01974 .01402 .01120 .00967 .00877 .00832 .00825 .00866 .00991 .01374 .32912

14 .51911 .03647 .01956 .01382 .01100 .00939 .00842 .00786 .00762 .00769 .00818 .00948 .01330 .32809

15 .51652 .03629 .01940 .01366 .01080 .00917 .00816 .00752 .00717 .00707 .00723 .00778 .00911 01293
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16 .51411 .03613 .01927 .01352 .01070 .00899 .00794 .00726 .00684 .00663 .00662 .00685 .00745 .00880
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17 .51186 .03598 .01915 .01340 .01050 .00885 .00777 .00706 .00659 .00631 .00619 .00625 .00653 .00716
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18 .50975 .03585 .01904 .01329 .01040 .00872 .00762 .00689 .00639 .00606 .00588 .00583 .00594 .00625
.00690 .00828 .01203 .32486

19 .50777 .03573 .01895 .01320 .01030 .00861 .00750 .00675 .00622 .00586 .00563 .00552 .00553 .00567
.00601 .00668 .00806 .01179 .32420

20 .50589 .03562 .01886 .01312 .01020 .00852 .00740 .00663 .00608 .00570 .00544 .00528 00522 .00527
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21 .50412 .03552 .01879 01304 .01020 .00843 .00730 .00852 .00597 .00556 .00528 .00509 .00499 .00497
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22 .50243 .03542 .01872 .01297 .01010 .00836 .00722 .00643 .00586 .00545 .00514 .00493 .00480 00473
00474 .00484 .00506 .00545 .00615 .00752 .01120 .32249

23 .50082 .03533 .01865 .01291 .01000 .00829 .00715 .00636 .00578 .00535 .00503 .00480 .00464 .00455
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24 .49929 .03525 .01859 .01286 .00996 .00823 .00709 .00629 .00570 .00526 .00493 .00469 .00451 .00439
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25 .49783 .03517 .01854 .01280 .00991 .00818 .00703 .00622 .00563 .00519 .00485 .00459 .00440 .00427
.00418 .00414 .00416 .00423 .00437 .00463 .00504 .00575 .00711 .01070 .32107

26 .49642 .03509 .01848 .01276 .00987 .00813 .00698 .00617 .00557 .00512 .00478 .00451 .00431 .00416
.00405 .00399 .00398 .00401 .00409 .00425 .00451 .00493 .00563 .00700 .01060 32065

27 .49508 .03502 .01843 .01271 .00982 .00808 .00693 .00612 .00552 .00506 .00471 .00444 .00423 .00406
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28 .49378 .03496 .01839 .01267 .00978 .00804 .00689 .00607 .00547 .00501 .00465 .00437 .00415 00398
.00386 .00376 .00371 .00368 .00370 .00375 .00386 .00403 .00430 .00472 .00544 .00679 .01030 .31986

29 .49253 .03489 .01835 .01263 .00974 .00800 .00685 .00603 .00542 .00496 .00460 .00431 .00409 .00391
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30 .49133 .03483 .01830 01260 .00971 .00797 .00681 .00599 .00538 .00492 .00455 .00426 .00403 .00385
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31 .49017 .03477 .01827 .01256 .00968 .00794 .00678 .00595 00534 .00488 00451 .00422 .00398 .00379
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.00652 .01000 .31882

32 .48905 .03472 .01823 .01253 .00965 .00791 .00675 .00592 .00531 .00484 .00447 .00417 .00394 00374
.00359 .00346 .00337 .00329 .00324 .00322 .00321 .00323 .00328 .00337 .00350 00369 .00397 .00440
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33 .48797 .03466 .01819 .01250 .00962 .00788 .00672 .00589 .00528 .00480 .00443 .00413 .00389 .00370
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34 .48692 .03461 .01816 .01247 .00959 .00785 .00669 .00586 .00525 .00477 .00440 .00410 00386 .00366
.00349 .00336 00325 .00316 .00310 .00305 .00303 .00302 .00303 .00306 .00312 00322 00335 .00355
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35 .48591 03456 .01813 01245 .00956 .00783 .00666 .00584 .00522 .00474 00437 .00407 00382 .00362
.00345 .00331 .00320 .00311 .00304 .00299 00295 .00293 .00293 .00295 .00299 00305 .00315 .00329
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36 .48492 .03451 .01810 .01242 .00954 .00780 .00664 .00581 .00519 .00472 .00434 .00404 .00379 .00358
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37 .48396 .03447 .01807 .01240 .00952 .00778 .00662 .00579 .00517 .00469 .00431 .00401 .00376 .00355
.00338 .00324 .00312 .00302 .00294 .00287 .00283 .00279 .00277 .00277 .00278 .00281 .00285 .00293
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38 .48303 .03442 .01804 .01237 .00950 .00776 .00660 .00577 .00515 .00467 .00429 .00398 .00373 .00352
.00335 .00320 .00308 .00298 .00290 .00283 .00277 .00273 .00271 .00269 .00269 .00271 .00274 .00279
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39 .48213 .03438 .01802 .01235 .00947 .00774 .00658 .00575 .00513 .00465 .00427 .00396 .00370 .00350
.00332 .00317 .00305 .00294 .00286 .00278 .00273 .00268 .00265 .00263 .00262 .00263 .00265 .00268
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40 .48125 .03434 .01799 .01233 .00945 .00772 .00656 .00573 .00511 .00463 .00424 .00393 .00368 .00347
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41 .48039 .03429 .01796 .01231 .00944 .00770 .00654 .00571 .00509 .00461 .00422 .00391 .00366 .00345
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43 .47874 .03422 .01792 .01227 .00940 .00767 .00650 .00567 .00505 .00457 .00419 .00387 .00362 .00340
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44 .47794 .03418 .01790 .01225 .00938 .00765 .00649 .00566 .00504 .00455 .00417 .00386 .00360 .00338
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45 .47717 .03414 .01787 .01223 .00937 .00763 .00647 .00564 .00502 .00454 .00415 .00384 .00358 .00337
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46 .47641 .03411 .01785 .01222 .00935 .00762 .00646 .00563 .00501 .00452 .00414 .00382 .00357 .00335
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47 .47567 .03407 .01783 .01220 .00934 .00761 .00645 .00562 .00499 .00451 .00412 .00381 00355 .00333
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48 .47495 .03404 .01781 .01218 .00932 .00759 .00643 .00560 .00498 .00450 .00411 .00379 .00353 .00332
.00313 .00297 .00284 .00272 .00262 .00253 .00245 .00238 .00232 .00227 .00223 .00220 .00217 .00215
.00213 .00212 .00212 .00213 .00214 .00216 .00219 .00223 .00229 .00236 .00245 00257 .00273 .00294
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49 47424 .03401 .01779 .01217 .00931 .00758 .00642 .00559 .00497 .00448 .00410 .00378 00352 .00330
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.00210 .00208 .00208 .00208 .00209 .00210 .00212 .00216 .00220 .00226 .00233 .00242 .00254 .00270
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50 .47354 .03397 .01777 .01215 .00930 .00757 .00641 .00558 .00495 .00447 .00408 .00377 00351 .00329
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.00206 .00205 .00204 .00203 .00204 .00205 .00206 .00209 .00212 00217 .00222 .00230 00239 .00251
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2	.60119 .39881
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4	.54224 .06479 .04726 .34571
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6	.51541 .05830 .03507 .02923 .03200 .32998
7	.50596 .05671 .03311 .02599 .02436 .02866 .32521
8	.49803 .05555 .03184 .02419 .02128 .02133 .02633 .32145
9	.49119 .05465 .03094 .02303 .01959 .01839 .01925 .02459 .31837
10	.48520 .05391 .03026 .02222 .01850 .01676 .01640 .01772 .02324 .31579
11	.47987 .05330 .02972 .02161 .01773 .01572 .01484 .01496 .01654 .02214 .31358
12	.47508 .05277 .02928 .02113 .01716 .01498 .01383 .01344 .01385 .01559 .02124 .31164
13	.47072 .05231 .02891 .02075 .01672 .01444 .01312 .01246 .01237 .01297 .01482 .02047 .30994
14	.46672 .05191 .02860 .02043 .01636 .01402 .01260 .01177 .01140 .01150 .01225 .01417 .01982 .30841
15	.46304 .05154 .02832 .02016 .01607 .01368 .01219 .01130 .01070 .01060 .01080 .01170 .01362 .01925 .30704
16	.45963 .05121 .02808 .01993 .01582 .01340 .01186 .01090 .01030 .00994 .00992 .01030 .01110 .01315 .01874 .30579
17	.45645 .05091 .02787 .01972 .01560 .01316 .01160 .01060 .00987 .00945 .00928 .00937 .00978 .01070 .01273 .01830 .30465
18	.45347 .05063 .02767 .01954 .01542 .01296 .01140 .01030 .00956 .00908 .00881 .00874 .00890 .00937 .01030 .01236 .01790 .30359
19	.45067 .05038 .02750 .01938 .01526 .01279 .01120 .01010 .00930 .00878 .00844 .00828 .00829 .00850 .00900 .00999 .01203 .01753 .30262
20	.44803 .05014 .02734 .01924 .01511 .01264 .01100 .00989 .00909 .00853 .00815 .00792 .00783 .00790 .00815 .00869 .00970 .01174 .01720 .30171
21	.44554 .04991 .02719 .01911 .01499 .01250 .01090 .00973 .00891 .00832 .00790 .00763 .00747 .00744 .00756 .00785 .00841 .00943 .01150 .01690 .30087
22	.44317 .04970 .02706 .01899 .01487 .01238 .01070 .00959 .00876 .00815 .00770 .00739 .00719 .00709 .00711 .00726 .00758 .00816 .00919 .01120 .01662 .30008
23	.44092 .04950 .02693 .01888 .01476 .01228 .01060 .00947 .00862 .00800 .00753 .00719 .00695 .00681 .00677 .00682 .00700 .00734 .00793 .00897 .01100 .01637 .29933
24	.43877 .04932 .02681 .01878 .01467 .01218 .01050 .00936 .00850 .00786 .00738 .00702 .00676 .00658 .00649 .00648 .00657 .00677 .00712 .00773 .00877 .01080 .01613 .29863
25	.43673 .04914 .02670 .01869 .01458 .01209 .01040 .00926 .00840 .00775 .00725 .00687 .00659 .00639 .00627 .00621 .00623 .00634 .00656 .00693 .00754 .00859 .01060 .01591 .29797
26	.43477 .04897 .02660 .01860 .01450 .01201 .01030 .00917 .00830 .00764 .00714 .00675 .00645 .00623 .00608 .00599 .00596 .00601 .00613 .00637 .00675 .00737 .00842 .01040 .01570 .29734
27	.43289 .04881 .02650 .01852 .01442 .01193 .01030 .00909 .00821 .00755 .00704 .00664 .00633 .00609 .00591 .00580 .00574 .00574 .00581 .00595 .00619 .00659 .00721 .00826 .01030 .01551 .29674
28	.43109 .04866 .02640 .01844 .01435 .01186 .01020 .00901 .00814 .00747 .00695 .00654 .00622 .00597 .00578 .00564 .00556 .00552 .00554 .00562 .00578 .00604 .00644 .00707 .00812 .01010 .01533 .29618
29	.42936 .04851 .02632 .01837 .01428 .01180 .01010 .00895 .00807 .00739 .00686 .00645 .00612 .00586 .00566 .00550 .00540 .00534 .00533 .00536 .00546 .00563 .00589 .00630 .00693 .00798 .00997 .01516 .29564
30	.42769 .04837 .02623 .01830 .01422 .01174 .01010 .00888 .00800 .00732 .00679 .00637 .00603 .00576 .00555 .00538 .00526 .00519 .00515 .00515 .00520 .00531 .00549 .00576 .00617 .00681 .00786 .00984 .01499 .29512
31	.42608 .04823 .02615 .01824 .01416 .01170 .01000 .00883 .00794 .00726 .00672 .00630 .00595 .00568 .00545 .00528 .00515 .00505 .00499 .00497 .00499 .00506 .00517 .00536 .00564 .00605 .00669 .00774 .00971 .01484 .29462
32	.42453 .04810 .02607 .01818 .01411 .01160 .00996 .00877 .00788 .00720 .00666 .00623 .00588 .00560 .00537 .00519 .00504 .00494 .00486 .00482 .00482 .00485 .00492 .00505 .00524 .00552 .00594 .00658 .00763 .00959 .01470 .29415
33	.42303 .04798 .02600 .01812 .01405 .01160 .00991 .00872 .00783 .00715 .00660 .00617 .00582 .00553 .00529 .00510 .00495 .00483 .00475 .00469 .00467 .00467 .00471 .00480 .00493 .00513 .00542 .00584 .00648 .00752 .00948 .01456 .29369
34	.42158 .04785 .02593 .01806 .01400 .01150 .00987 .00868 .00778 .00710 .00655 .00611 .00576 .00546 .00522 .00503 .00487 .00474 .00465 .00458 .00454 .00452 .00454 .00459 .00468 .00482 .00502 .00532 .00574 .00638 .00742 .00937 .01443 .29326
35	.42018 .04774 .02586 .01801 .01396 .01150 .00982 .00863 .00774 .00705 .00650 .00606 .00570 .00541 .00516 .00496 .00479 .00466 .00456 .00448 .00442 .00440 .00439 .00442 .00448 .00458 .00472 .00493 .00522 .00565 .00629 .00733 .00927 .01430 .29284

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36   .41882 .04762 .02580 .01796 .01391 .01140 .00978 .00859 .00770 .00701 .00646 .00601 .00565 .00535
      .00510 .00490 .00473 .00459 .00447 .00439 .00432 .00428 .00427 .00427 .00431 .00437 .00448 .00463
      .00484 .00514 .00556 .00620 .00724 .00918 .01418 .29243

37   .41750 .04751 .02574 .01791 .01387 .01140 .00974 .00855 .00766 .00697 .00642 .00597 .00561 .00530
      .00505 .00484 .00467 .00452 .00440 .00431 .00424 .00419 .00416 .00415 .00416 .00421 .00428 .00439
      .00454 .00475 .00505 .00548 .00612 .00716 .00909 .01407 .29204

38   .41621 .04741 .02567 .01786 .01383 .01140 .00971 .00852 .00762 .00693 .00638 .00593 .00556 .00526
      .00500 .00479 .00461 .00446 .00434 .00424 .00416 .00410 .00406 .00404 .00404 .00406 .00411 .00419
      .00430 .00446 .00468 .00498 .00541 .00605 .00708 .00900 .01396 .29166

39   .41497 .04730 .02562 .01782 .01379 .01130 .00967 .00848 .00759 .00689 .00634 .00589 .00552 .00521
      .00496 .00474 .00456 .00441 .00428 .00417 .00409 .00402 .00397 .00394 .00393 .00394 .00397 .00402
      .00410 .00422 .00438 .00460 .00490 .00533 .00597 .00700 .00891 .01386 .29130

40   .41376 .04720 .02556 .01777 .01375 .01130 .00964 .00845 .00755 .00686 .00630 .00585 .00548 .00518
      .00492 .00470 .00451 .00435 .00422 .00411 .00402 .00395 .00389 .00386 .00383 .00383 .00385 .00388
      .00394 .00402 .00414 .00431 .00453 .00484 .00527 .00590 .00693 .00883 .01376 .29095

41   .41258 .04711 .02551 .01773 .01372 .01130 .00961 .00842 00752 .00683 .00627 .00582 .00545 .00514
      .00488 .00466 .00447 .00431 .00417 .00406 .00396 .00389 .00382 .00378 .00375 .00374 .00374 .00376
      .00380 .00386 .00395 .00407 .00424 .00446 .00477 .00520 .00584 .00686 .00876 .01366 .29060

42   .41143 .04701 .02545 .01769 .01368 .01120 .00958 .00839 .00749 .00680 .00624 .00579 .00542 .00510
      .00484 .00462 .00443 .00426 .00413 .00401 .00391 .00383 .00376 .00371 .00367 .00365 .00364 .00365
      .00368 .00372 .00379 .00388 .00401 .00417 .00440 .00471 .00514 .00577 .00679 .00868 .01357 .29027

43   .41032 .04692 .02540 .01765 .01365 .01120 .00955 .00836 .00746 .00677 .00621 .00576 .00538 .00507
      .00481 .00458 .00439 .00422 .00408 .00396 .00386 .00378 .00370 .00365 .00361 .00358 .00356 .00356
      .00357 .00360 .00365 .00372 .00381 .00394 .00411 .00434 .00465 .00508 .00571 .00673 .00861 .01348
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44   .40923 .04683 .02535 .01762 .01362 .01120 .00952 .00833 .00744 .00674 .00618 .00573 .00535 .00504
      .00477 ..00455 .00435 .00419 .00404 .00392 .00382 .00373 .00365 .00359 .00354 .00351 .00349 .00348
      .00348 .00350 .00353 .00358 .00365 .00375 .00388 .00405 .00428 .00459 .00502 .00566 .00667 00855
      .01339 .28964

45   .40817 .04674 .02530 .01758 .01358 .01110 .00949 .00830 .00741 .00671 .00616 .00570 .00533 .00501
      .00474 .00452 .00432 .00415 .00401 .00388 .00377 .00368 .00360 .00354 .00349 .00345 .00342 .00340
      .00340 .00340 .00343 .00346 .00352 .00359 .00369 .00383 .00400 .00423 .00454 .00497 .00560 .00661
      .00848 .01331 .28934

46   .40713 .04666 .02526 .01754 .01355 .01110 .00947 .00828 .00739 .00669 .00613 .00568 .00530 .00498
      .00472 .00449 .00429 .00412 00397 .00385 .00374 .00364 .00356 .00349 .00344 .00339 .00336 .00333
      .00332 .00332 .00333 .00336 .00340 .00346 .00354 .00364 .00377 .00395 .00418 .00449 .00492 .00555
      .00655 .00842 .01323 .28904

47   .40612 .04657 .02521 .01751 .01353 .01110 .00944 .00826 .00736 .00666 .00611 .00565 .00528 .00496
       00469 .00446 .00426 .00409 .00394 .00381 .00370 .00360 .00352 .00345 .00339 00334 .00330 .00327
      .00326 .00325 .00325 .00327 .00330 .00334 .00340 .00348 .00358 .00372 .00390 .00413 .00444 .00487
      .00550 .00650 .00836 .01315 .28876

48   .40513 .04649 .02516 .01748 .01350 .01110 .00942 .00823 .00734 .00664 .00608 .00563 .00525 .00493
      .00466 .00443 .00423 .00406 .00391 .00378 .00367 .00357 .00348 .00341 .00334 00329 .00325 .00322
      .00320 .00318 .00318 .00319 .00321 .00324 .00328 .00335 .00343 .00353 .00367 .00385 .00408 .00439
      .00482 .00545 .00645 .00830 .01308 .28848

49   .40416 .04641 .02512 .01744 .01347 .01100 .00939 .00821 .00732 .00662 .00606 .00561 .00523 .00491
      .00464 .00441 .00421 .00403 .00388 .00375 .00363 .00353 .00345 .00337 .00330 .00325 .00320 .00317
      .00314 .00312 .00312 .00312 .00313 .00315 .00318 .00323 .00330 .00338 .00349 .00362 .00380 .00403
      .00434 .00477 .00540 .00640 .00824 .01300 .28821

50   .40322 .04633 .02508 .01741 .01344 .01100 .00937 .00819 .00730 .00660 .00604 .00559 .00521 .00489
      .00462 00438 .00418 .00401 .00385 .00372 .00360 .00350 .00341 .00333 .00327 .00321 .00316 .00312
      .00309 .00307 .00306 .00305 .00306 .00307 .00309 .00313 .00318 .00325 .00333 .00344 .00358 .00376
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4 .51451 .08440 .06266 .33843

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6 .47989 .07501 .04625 .03883 .04248 .31755

7 .46779 .07260 .04353 .03446 .03239 .03802 .31122

8 .45767 .07079 .04173 .03202 .02829 .02837 .03490 .30623

9 .44900 .06935 .04042 .03043 .02601 .02446 .02560 .03257 .30216

10 .44143 .06817 .03942 .02929 .02453 .02229 .02184 .02355 .03075 .29874

11 .43471 .06716 .03861 .02843 .02348 .02088 .01974 .01991 .02198 .02929 .29581

12 .42869 .06629 .03795 .02774 .02268 .01989 .01839 .01789 .01843 .02071 .02807 .29326

13 .42323 .06552 .03738 .02718 .02206 .01914 .01744 .01658 .01647 .01726 .01968 .02705 .29101

14 .41824 .06483 .03689 .02672 .02156 .01856 .01673 .01566 .01520 .01534 .01630 .01881 .02616 .28900

15 .41366 .06421 .03646 .02632 .02114 .01809 .01617 .01498 .01431 .01411 .01443 .01550 .01807 .02540 .28718

16 .40941 .06364 .03607 .02597 .02078 .01770 .01572 .01444 .01364 .01324 .01322 .01366 .01482 .01743 .02472 .28553

17 .40547 .06312 .03573 .02566 .02047 .01737 .01535 .01401 .01312 .01259 .01237 .01248 .01302 .01424 .01687 .02412 .28403

18 .40178 .06264 .03542 .02539 .02020 .01708 .01504 .01365 .01270 .01208 .01173 .01160 .01186 .01246 .01373 .01638 .02358 .28264

19 .39833 .06219 .03513 .02514 .01996 .01683 .01477 .01335 .01236 .01170 .01120 .01100 .01100 .01130 .01198 .01328 .01594 .02309 .28136

20 .39508 .06177 .03487 .02492 .01975 .01661 .01454 .01310 .01207 .01130 .01080 .01050 .01040 .01050 .01090 .01160 .01288 .01554 .02265 .28016

21 .39201 .06137 .03462 .02471 .01956 .01642 .01433 .01287 .01182 .01110 .01050 .01020 .00995 .00992 .01010 .01040 .01120 .01252 .01518 .02224 .27905

22 .38910 .06100 .03439 .02452 .01938 .01624 .01415 .01268 .01160 .01080 .01020 .00983 .00957 .00945 .00948 .00967 .01010 .01080 .01219 .01485 .02187 .27801

23 .38635 .06065 .03418 .02435 .01922 .01608 .01399 .01250 .01140 .01060 .01000 .00956 .00926 .00908 .00902 .00909 .00932 .00977 .01050 .01190 .01455 .02152 .27703

24 .38372 .06032 .03398 .02419 .01907 .01594 .01384 .01235 .01130 .01040 .00980 .00934 .00900 .00877 .00865 .00864 .00875 .00901 .00948 .01030 .01160 .01428 .02120 .27611

25 .38122 .06001 .03379 .02404 .01893 .01581 .01370 .01221 .01110 .01030 .00963 .00914 .00877 .00851 .00835 .00827 .00830 .00844 .00873 .00921 .01000 .01140 .01402 .02091 .27524

26 .37884 .05970 .03361 .02389 .01881 .01568 .01358 .01208 .01100 .01010 .00947 .00896 .00858 .00829 .00809 .00798 .00794 .00800 .00817 .00847 .00898 .00979 .01120 .01379 .02063 .27441

27 .37655 .05942 .03344 .02376 .01869 .01557 .01347 .01196 .01080 .00999 .00933 .00881 .00841 .00810 .00788 .00773 .00765 .00765 .00773 .00792 .00824 .00876 .00958 .01090 .01357 .02037 .27363

28 .37436 .05914 .03328 .02364 .01858 .01546 .01336 .01186 .01070 .00987 .00920 .00868 .00826 .00793 .00769 .00751 .00740 .00736 .00739 .00749 .00770 .00803 .00856 .00938 .01080 .01336 .02012 .27289

29 .37226 .05888 .03313 .02352 .01847 .01536 .01327 .01176 .01060 .00977 .00909 .00855 .00813 .00779 .00753 .00733 .00719 .00712 .00710 .00715 .00727 .00749 .00784 .00837 .00920 .01060 .01317 .01989 .27218

30 .37024 .05863 .03298 .02340 .01837 .01527 .01317 .01170 .01050 .00967 .00899 .00844 .00801 .00766 .00738 .00717 .00701 .00691 .00686 .00687 .00693 .00707 .00731 .00766 .00820 .00903 .01040 .01299 .01968 .27150

31 .36829 .05838 .03284 .02330 .01828 .01518 .01309 .01160 .01050 .00958 .00889 .00834 .00790 .00754 .00725 .00703 .00685 .00673 .00665 .00663 .00665 .00674 .00689 .00713 .00750 .00804 .00887 .01020 .01282 .01947 .27085

32 .36641 .05815 .03271 .02319 .01819 .01510 .01301 .01150 .01040 .00950 .00881 .00825 .00780 .00743 .00714 .00690 .00671 .00657 .00648 .00643 .00642 .00646 .00656 .00672 .00697 .00734 .00789 .00873 .01010 .01266 .01928 .27023

33 .36460 .05792 .03258 .02310 .01811 .01502 .01294 .01140 .01030 .00942 .00873 .00817 .00771 .00734 .00703 .00679 .00659 .00644 .00632 .00625 .00622 .00623 .00628 .00639 .00656 .00682 .00720 .00775 .00859 .00995 .01251 .01909 .26963

34 .36286 .05770 .03246 .02300 .01803 .01495 .01286 .01140 .01020 .00935 .00865 .00809 .00763 .00725 .00694 .00668 .00648 .00631 .00619 .00610 .00604 .00603 .00605 .00612 .00624 .00642 .00669 .00707 .00762 .00846 .00982 .01236 .01891 .26906

35 .36116 .05749 .03234 .02291 .01795 .01488 .01280 .01130 .01020 .00928 .00858 .00802 .00755 .00717 .00685 .00659 .00638 .00620 .00607 .00596 .00589 .00586 .00586 .00589 .00597 .00610 .00628 .00656 .00694 .00750 .00834 .00970 .01223 .01875 .26850

36	.35953	.05729	.03222	.02283	.01787	.01481	.01273	.01120	.01010	.00922	.00852	.00795	.00748	.00710
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	.00644	.00683	.00739	.00822	.00958	.01210	.01859	.26797						
37	.35794	.05709	.03211	.02274	.01780	.01475	.01267	.01120	.01000	.00916	.00846	.00789	.00742	.00703
	.00670	.00643	.00620	.00601	.00586	.00574	.00564	.00558	.00554	.00553	.00555	.00560	.00570	.00584
	.00604	.00632	.00672	.00728	.00811	.00946	.01197	.01843	.26746					
38	.35640	.05690	.03200	.02266	.01774	.01469	.01262	.01110	.00999	.00910	.00840	.00783	.00736	.00696
	.00663	.00636	.00613	.00593	.00577	.00564	.00554	.00546	.00541	.00538	.00538	.00541	.00548	.00558
	.00573	.00593	.00622	.00661	.00717	.00801	.00936	.01186	.01829	.26697				
39	.35491	.05671	.03190	.02258	.01767	.01463	.01256	.01110	.00993	.00905	.00835	.00777	.00730	.00690
	.00657	.00629	.00606	.00586	.00569	.00555	.00544	.00535	.00529	.00525	.00524	.00525	.00529	.00536
	.00547	.00562	.00583	.00612	.00651	.00708	.00791	.00925	.01174	.01815	.26649			
40	.35346	.05653	.03180	.02251	.01761	.01457	.01251	.01100	.00988	.00900	.00830	.00772	.00725	.00685
	.00651	.00623	.00599	.00579	.00562	.00547	.00535	.00526	.00519	.00514	.00511	.00511	.00513	.00517
	.00525	.00536	.00552	.00573	.00602	.00642	.00699	.00782	.00916	.01160	.01801	.26603		
41	.35205	.05635	.03170	.02244	.01755	.01452	.01246	.01100	.00984	.00895	.00825	.00767	.00719	.00680
	.00646	.00617	.00593	.00572	.00555	.00540	.00528	.00517	.00510	.00504	.00500	.00498	.00498	.00501
	.00506	.00514	.00526	.00542	.00564	.00593	.00633	.00690	.00773	.00906	.01150	.01788	.26558	
42	.35068	.05618	.03161	.02237	.01749	.01447	.01241	.01090	.00979	.00891	.00820	.00763	.00715	.00675
	.00641	.00612	.00587	.00566	.00548	.00533	.00520	.00510	.00501	.00494	.00490	.00487	.00486	.00487
	.00490	.00496	.00505	.00517	.00533	.00555	.00585	.00625	.00681	.00764	.00897	.01140	.01776	.26515
43	.34935	.05601	.03151	.02230	.01744	.01442	.01236	.01090	.00975	.00887	.00816	.00758	.00710	.00670
	.00636	.00607	.00582	.00561	.00543	.00527	.00514	.00503	.00493	.00486	.00480	.00477	.00475	.00474
	.00476	.00480	.00486	.00495	.00508	.00525	.00547	.00577	.00617	.00673	.00756	.00889	.01130	.01764
	.26473													
44	.34805	.05585	.03142	.02223	.01738	.01437	.01232	.01080	.00971	.00883	.00812	.00754	.00706	.00666
	.00632	.00602	.00577	.00556	.00537	.00521	.00508	.00496	.00486	.00478	.00472	.00468	.00465	.00463
	.00464	.00466	.00470	.00477	.00487	.00500	.00517	.00539	.00569	.00609	.00666	.00749	.00881	.01120
	.01752	.26432												
45	.34679	.05569	.03134	.02217	.01733	.01433	.01228	.01080	.00967	.00879	.00808	.00750	.00702	.00662
	.00627	.00598	.00573	.00551	.00532	.00516	.00502	.00490	.00480	.00471	.00465	.00459	.00456	.00453
	.00453	.00454	.00457	.00461	.00469	.00479	.00492	.00509	.00532	.00562	.00602	.00658	.00741	.00873
	.01120	.01741	.26392											
46	.34555	.05554	.03125	.02211	.01728	.01428	.01224	.01080	.00963	.00875	.00804	.00746	.00698	.00658
	.00623	.00594	.00568	.00546	.00527	.00511	.00497	.00484	.00474	.00465	.00458	.00452	.00447	.00444
	.00443	.00443	.00444	.00448	.00453	.00461	.00471	.00484	.00502	.00525	.00555	.00595	.00652	.00734
	.00865	.01110	.01730	.26354										
47	.34435	.05539	.03117	.02205	.01723	.01424	.01220	.01070	.00959	.00871	.00801	.00743	.00695	.00654
	.00619	.00590	.00564	.00542	.00523	.00506	.00492	.00479	.00468	.00459	.00451	.00445	.00440	.00436
	.00434	.00433	.00434	.00436	.00439	.00445	.00453	.00463	.00477	.00495	.00518	.00548	.00589	.00645
	.00727	.00858	.01100	.01720	.26317									
48	.34317	.05524	.03109	.02199	.01718	.01419	.01216	.01070	.00956	.00868	.00797	.00739	.00691	.00650
	.00616	.00586	.00560	.00538	.00519	.00502	.00487	.00474	.00463	.00453	.00445	.00439	.00433	.00429
	.00426	.00424	.00424	.00425	.00427	.00432	.00438	.00446	.00456	.00470	.00488	.00511	.00542	.00582
	.00638	.00720	.00851	.01090	.01710	.26280								
49	.34202	.05510	.03101	.02193	.01713	.01415	.01212	.01060	.00952	.00865	.00794	.00736	.00688	.00647
	.00612	.00583	.00557	.00534	.00515	.00498	.00483	.00470	.00458	.00448	.00440	.00433	.00427	.00422
	.00419	.00416	.00415	.00415	.00417	.00420	.00424	.00430	.00439	.00450	.00464	.00482	.00505	.00536
	.00576	.00632	.00714	.00844	.01080	.01700	.26245							
50	.34090	.05495	.03093	.02188	.01709	.01411	.01208	.01060	.00949	.00861	.00791	.00733	.00685	.00644
	.00609	.00579	.00553	.00531	.00511	.00494	.00479	.00465	.00454	.00444	.00435	.00427	.00421	.00416
	.00412	.00409	.00407	.00407	.00407	.00409	.00412	.00417	.00424	.00432	.00443	.00458	.00476	.00499
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2	.58267 .41733
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5	.46457 .09481 .06322 .06155 .31586
6	.44657 .08998 .05706 .04830 .05279 .30530
7	.43214 .08658 .05348 .04280 .04035 .04720 .29743
8	.42014 .08399 .05107 .03967 .03523 .03536 .04329 .29125
9	.40991 .08190 .04930 .03760 .03234 .03050 .03189 .04036 .28620
10	.40100 .08016 .04792 .03610 .03045 .02777 .02723 .02933 .03807 .28197
11	.39313 .07867 .04679 .03495 .02909 .02599 .02462 .02484 .02735 .03622 .27835
12	.38610 .07737 .04584 .03403 .02806 .02472 .02293 .02232 .02299 .02577 .03469 .27520
13	.37975 .07621 .04503 .03326 .02723 .02376 .02172 .02069 .02055 .02152 .02447 .03339 .27241
14	.37397 .07518 .04433 .03262 .02656 .02300 .02081 .01953 .01897 .01915 .02032 .02338 .03228 .26993
15	.36866 .07424 .04370 .03206 .02600 .02238 .02009 .01865 .01785 .01761 .01800 .01931 .02244 .03131 .26769
16	.36377 .07338 .04314 .03157 .02551 .02186 .01951 .01797 .01701 .01652 .01650 .01705 .01846 .02164 .03046 .26566
17	.35923 .07259 .04263 .03113 .02509 .02142 .01903 .01742 .01635 .01570 .01544 .01558 .01624 .01772 .02093 .02970 .26381
18	.35501 .07185 .04216 .03074 .02472 .02104 .01862 .01696 .01581 .01506 .01464 .01454 .01480 .01554 .01708 .02031 .02901 .26210
19	.35105 .07117 .04173 .03039 .02439 .02071 .01826 .01657 .01537 .01455 .01402 .01376 .01378 .01413 .01493 .01651 .01975 .02840 .26052
20	.34734 .07053 .04134 .03007 .02409 .02041 .01795 .01623 .01500 .01412 .01352 .01316 .01302 .01313 .01355 .01440 .01601 .01925 .02784 .25906
21	.34385 .06993 .04097 .02977 .02381 .02014 .01767 .01594 .01468 .01376 .01311 .01267 .01243 .01239 .01257 .01304 .01393 .01556 .01879 .02732 .25769
22	.34054 .06936 .04063 .02949 .02357 .01990 .01743 .01568 .01440 .01345 .01276 .01226 .01195 .01180 .01183 .01207 .01258 .01350 .01515 .01838 .02685 .25641
23	.33741 .06883 .04030 .02924 .02334 .01968 .01721 .01545 .01415 .01318 .01246 .01192 .01160 .01130 .01130 .01140 .01160 .01218 .01312 .01478 .01800 .02641 .25521
24	.33444 .06832 .04000 .02900 .02312 .01948 .01701 .01524 .01393 .01294 .01220 .01160 .01120 .01090 .01080 .01080 .01090 .01120 .01181 .01278 .01444 .01765 .02601 .25408
25	.33162 .06784 .03971 .02877 .02293 .01929 .01682 .01505 .01373 .01273 .01197 .01140 .01090 .01060 .01040 .01030 .01040 .01050 .01090 .01150 .01246 .01413 .01733 .02563 .25301
26	.32892 .06738 .03944 .02856 .02274 .01912 .01665 .01488 .01356 .01254 .01176 .01120 .01070 .01030 .01010 .00996 .00992 .00999 .01020 .01060 .01120 .01217 .01384 .01703 .02528 .25200
27	.32635 .06694 .03918 .02836 .02257 .01895 .01649 .01472 .01339 .01237 .01160 .01100 .01050 .01010 .00983 .00965 .00955 .00956 .00966 .00989 .01030 .01090 .01190 .01358 .01675 .02495 .25104
28	.32388 .06652 .03893 .02818 .02241 .01880 .01635 .01457 .01324 .01222 .01140 .01080 .01030 .00989 .00959 .00938 .00925 .00919 .00923 .00936 .00961 .01000 .01070 .01170 .01333 .01649 .02465 .25013
29	.32152 .06611 .03870 .02800 .02225 .01866 .01621 .01444 .01311 .01208 .01130 .01060 .01010 .00970 .00938 .00915 .00898 .00889 .00887 .00893 .00908 .00935 .00977 .01040 .01140 .01310 .01625 .02435 .24926
30	.31926 .06573 .03847 .02783 .02211 .01853 .01608 .01431 .01298 .01195 .01110 .01050 .00995 .00953 .00920 .00894 .00875 .00863 .00857 .00858 .00866 .00883 .00911 .00955 .01020 .01120 .01289 .01602 .02408 .24843
31	.31708 .06536 .03826 .02766 .02197 .01840 .01597 .01420 .01286 .01183 .01100 .01030 .00981 .00938 .00904 .00876 .00855 .00840 .00831 .00828 .00831 .00841 .00860 .00890 .00934 .01000 .01100 .01269 .01581 .02382 .24764
32	.31498 .06500 .03805 .02751 .02184 .01828 .01585 .01409 .01275 .01170 .01090 .01020 .00969 .00925 .00889 .00860 .00837 .00821 .00809 .00803 .00802 .00807 .00819 .00839 .00869 .00915 .00982 .01080 .01250 .01561 .02357 .24688
33	.31296 .06465 .03785 .02736 .02171 .01817 .01575 .01398 .01265 .01160 .01080 .01010 .00957 .00912 .00875 .00846 .00822 .00803 .00789 .00781 .00777 .00778 .00785 .00798 .00819 .00851 .00897 .00964 .01070 .01232 .01541 .02334 .24615
34	.31101 .06432 .03766 .02722 .02159 .01806 .01565 .01389 .01255 .01150 .01070 .01000 .00946 .00901 .00863 .00832 .00807 .00788 .00772 .00762 .00755 .00753 .00756 .00764 .00779 .00801 .00833 .00880 .00948 .01050 .01215 .01523 .02312 .24545
35	.30913 .06400 .03747 .02708 .02148 .01796 .01555 .01379 .01246 .01140 .01060 .00991 .00936 .00890 .00852 .00821 .00795 .00774 .00757 .00745 .00736 .00732 .00731 .00736 .00745 .00761 .00784 .00817 .00864 .00932 .01030 .01199 .01506 .02290 .24477

36 .30731 .06368 .03730 .02695 .02137 .01786 .01546 .01371 .01238 .01130 .01050 .00983 .00927 .00880
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37 .30555 .06338 .03712 .02682 .02126 .01777 .01537 .01362 .01229 .01130 .01040 .00974 .00918 .00871
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38 .30384 .06309 .03696 .02670 .02116 .01768 .01529 .01354 .01222 .01120 .01030 .00966 .00910 .00863
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39 .30219 .06281 .03679 .02658 .02106 .01759 .01521 .01347 .01214 .01110 .01030 .00959 .00902 00855
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40 .30059 .06253 .03664 .02646 .02097 .01751 .01513 .01340 .01207 .01100 .01020 .00952 .00895 .00848
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46 .29187 .06103 .03579 .02584 .02046 .01707 .01473 .01302 .01170 .01070 .00985 .00917 .00860 .00812
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47 .29055 .06080 .03566 .02575 .02038 .01700 .01467 .01296 .01170 .01060 .00980 .00912 .00855 .00807
.00765 .00730 .00699 .00673 .00650 .00630 .00612 .00597 .00584 .00573 .00563 .00555 .00549 .00545
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50 .28677 .06015 .03529 .02548 .02017 .01682 .01450 .01281 .01150 01050 .00966 .00898 .00841 .00793
.00752 .00716 .00685 .00658 .00634 .00613 .00595 .00579 .00565 .00553 .00542 .00533 00526 .00519
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7	.39911 .09868 .06293 .05097 .04823 .05617 .28391
8	.38551 .09518 .05984 .04711 .04209 .04227 .05145 .27656
9	.37397 .09234 .05753 .04452 .03857 .03648 .03812 .04790 .27057
10	.36397 .08996 .05570 .04262 .03624 .03319 .03259 .03504 .04513 .26555
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13	.34031 .08452 .05183 .03894 .03221 .02827 .02594 .02476 .02461 .02574 .02917 .03946 .25425
14	.33392 .08309 .05086 .03809 .03134 .02732 .02482 .02336 .02272 .02293 .02429 .02784 .03811 .25131
15	.32807 .08179 .05000 .03735 .03061 .02654 .02394 .02229 .02137 .02109 .02155 .02308 .02671 .03693 .24868
16	.32270 .08060 .04923 .03669 .02998 .02588 .02321 .02145 .02035 .01979 .01977 .02040 .02205 .02574 .03589 .24628
17	.31773 .07950 .04852 .03610 .02943 .02531 .02260 .02076 .01954 .01880 .01850 .01866 .01943 .02116 .02488 .03497 .24410
18	.31312 .07849 .04788 .03557 .02893 .02482 .02208 .02019 .01889 .01802 .01754 .01743 .01773 .01859 .02038 .02412 .03414 .24209
19	.30881 .07754 .04728 .03509 .02849 .02438 .02163 .01970 .01834 .01739 .01679 .01649 .01652 .01692 .01786 .01969 .02344 .03339 .24024
20	.30477 .07666 .04673 .03465 .02809 .02399 .02123 .01928 .01788 .01687 .01618 .01576 .01561 .01574 .01622 .01721 .01908 .02283 .03270 .23851
21	.30098 .07583 .04622 .03424 .02772 .02364 .02088 .01891 .01747 .01642 .01567 .01517 .01489 .01485 .01506 .01560 .01664 .01853 .02228 .03208 .23691
22	.29741 .07504 .04573 .03386 .02739 .02332 .02056 .01858 .01712 .01604 .01524 .01468 .01432 .01415 .01418 .01446 .01506 .01613 .01804 .02178 .03151 .23541
23	.29403 .07431 .04528 .03351 .02708 .02303 .02027 .01828 .01681 .01570 .01487 .01426 .01384 .01358 .01350 .01360 .01393 .01457 .01567 .01759 .02132 .03098 .23400
24	.29083 .07361 .04486 .03317 .02679 .02276 .02000 .01801 .01653 .01541 .01455 .01390 .01343 .01311 .01295 .01293 .01309 .01346 .01413 .01525 .01718 .02089 .03048 .23268
25	.28779 .07294 .04445 .03286 .02652 .02251 .01976 .01777 .01628 .01514 .01426 .01359 .01308 .01272 .01249 .01239 .01243 .01263 .01304 .01373 .01486 .01680 .02050 .03003 .23143
26	.28490 .07231 .04407 .03257 .02626 .02228 .01953 .01755 .01605 .01490 .01401 .01331 .01278 .01238 .01210 .01194 .01190 .01198 .01222 .01265 .01336 .01451 .01645 .02014 .02960 .23024
27	.28214 .07170 .04370 .03229 .02603 .02206 .01933 .01734 .01584 .01469 .01378 .01307 .01251 .01208 .01177 .01160 .01150 .01150 .01160 .01184 .01230 .01303 .01419 .01613 .01980 .02920 .22912
28	.27951 .07113 .04336 .03203 .02580 .02185 .01913 .01715 .01565 .01449 .01357 .01285 .01227 .01182 .01150 .01120 .01110 .01100 .01110 .01120 .01150 .01198 .01273 .01389 .01583 .01949 .02882 .22805
29	.27699 .07057 .04303 .03178 .02559 .02166 .01895 .01697 .01547 .01431 .01338 .01265 .01206 .01160 .01120 .01100 .01080 .01070 .01060 .01070 .01090 .01120 .01170 .01244 .01362 .01555 .01919 .02847 .22703
30	.27457 .07004 .04271 .03154 .02539 .02148 .01878 .01680 .01531 .01414 .01321 .01247 .01187 .01140 .01100 .01070 .01050 .01030 .01030 .01030 .01040 .01060 .01090 .01140 .01218 .01336 .01529 .01891 .02814 .22606
31	.27225 .06953 .04241 .03131 .02520 .02131 .01862 .01665 .01515 .01398 .01305 .01230 .01170 .01120 .01080 .01050 .01020 .01010 .00996 .00993 .00996 .01010 .01030 .01060 .01120 .01194 .01312 .01505 .01865 .02782 .22514
32	.27003 .06904 .04212 .03109 .02501 .02114 .01847 .01650 .01501 .01384 .01290 .01215 .01150 .01100 .01060 .01030 .01000 .00983 .00970 .00962 .00961 .00967 .00981 .01000 .01040 .01090 .01170 .01289 .01482 .01840 .02752 .22425
33	.26788 .06857 .04183 .03088 .02484 .02099 .01832 .01636 .01487 .01370 .01277 .01201 .01140 .01090 .01050 .01010 .00983 .00962 .00946 .00936 .00931 .00933 .00941 .00956 .00981 .01020 .01070 .01150 .01268 .01460 .01817 .02723 .22340
34	.26582 .06812 .04157 .03068 .02467 .02084 .01818 .01623 .01474 .01357 .01264 .01187 .01120 .01070 .01030 .00995 .00966 .00943 .00925 .00913 .00905 .00903 .00906 .00916 .00933 .00959 .00997 .01050 .01130 .01248 .01440 .01795 .02696 .22258
35	.26383 .06768 .04130 .03048 .02451 .02070 .01805 .01611 .01462 .01345 .01252 .01175 .01110 .01060 .01020 .00980 .00950 .00926 .00907 .00892 .00883 .00877 .00877 .00882 .00893 .00911 .00938 .00977 .01030 .01110 .01230 .01420 .01774 .02671 .22179

36	.26191	.06725	.04105	.03030	.02435	.02056	.01793	.01599	.01451	.01334	.01240	.01160	.01100	.01050
	.01000	.00966	.00936	.00910	.00890	.00874	.00862	.00855	.00852	.00853	.00860	.00872	00891	.00919
	.00959	.01010	.01090	.01212	.01402	.01754	.02646	.22104						
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	.00991	.00954	.00922	.00896	.00875	.00857	.00844	.00835	.00830	.00828	.00831	.00839	.00853	.00873
	.00901	.00941	.00997	.01080	.01195	.01385	.01735	.02622	.22031					
38	.25825	.06644	.04058	.02994	.02406	.02031	.01769	.01577	.01430	.01313	.01220	.01140	.01080	.01030
	.00980	.00942	.00910	.00883	.00861	.00842	.00828	.00817	.00810	.00806	.00807	.00811	.00820	.00834
	.00855	.00885	.00925	.00981	.01060	.01179	.01368	.01717	.02600	.21961				
39	.25652	.06606	.04035	.02978	.02393	.02019	.01758	.01567	.01420	.01304	.01210	.01130	.01070	.01020
	.00970	.00931	.00899	.00871	.00848	.00829	.00813	.00801	.00792	.00787	.00785	.00786	.00792	.00802
	.00817	.00839	.00869	.00910	.00966	.01050	.01160	.01352	.01700	.02578	.21893			
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	.00758	.00769	.00786	.00809	.00840	.00882	.00939	.01020	.01140	.01323	.01667	.02538	.21764	
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	.00734	.00742	.00755	.00772	.00795	.00827	.00869	.00926	.01010	.01120	.01309	.01652	.02518	.21702
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	.00906	.00866	.00831	.00801	.00774	.00751	.00731	.00713	.00698	.00685	.00674	.00666	.00659	.00654
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	.00627	.00624	.00622	.00622	.00625	.00629	.00635	.00644	.00656	.00672	.00692	.00717	.00750	.00793
	.00851	.00930	.01050	.01227	.01561	.02402	.21320							
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25 .21594 .07663 .05065 .03909 .03244 .02810 .02504 .02279 .02107 .01975 .01871 .01791 .01731 .01687
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27 .21004 .07472 .04942 .03813 .03162 .02736 .02435 .02211 .02040 .01906 .01801 .01717 .01650 .01599
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29 .20470 .07300 .04830 .03727 .03089 .02670 .02374 02153 .01983 01849 .01742 .01655 .01586 01530
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30 .20221 .07219 .04779 03687 .03056 .02640 .02346 .02126 .01957 .01823 .01715 .01628 .01558 .01500
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31 .19983 .07142 04729 03649 .03023 .02612 .02319 02101 .01932 .01799 .01691 01604 .01532 01473
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32 19755 .07068 .04681 .03612 .02993 .02585 .02294 02077 .01909 01776 .01669 01581 01508 .01449
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34 .19326 .06928 .04592 .03544 .02935 .02534 .02248 02034 .01867 .01735 .01627 .01539 .01466 .01405
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35 .19124 .06863 .04550 .03511 .02908 .02510 02226 .02013 .01848 .01716 .01609 .01520 .01447 .01385
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36	.18930	.06799	.04509	.03480	.02882	.02487	.02206	.01994	.01829	.01698	.01591	.01503	.01429	.01367
	.01315	.01271	.01235	.01204	.01180	.01160	.01150	.01140	.01130	.01140	.01140	.01160	.01183	.01217
	.01266	.01334	.01431	.01575	.01805	.02225	.03264	.17775						
37	.18743	.06738	.04470	.03450	.02857	.02465	.02186	.01975	.01812	.01681	.01574	.01486	.01412	.01350
	.01298	.01253	.01216	.01184	.01160	.01140	.01120	.01110	.01100	.01100	.01110	.01120	.01130	.01160
	.01193	.01242	.01310	.01408	.01551	01781	.02199	.03232	.17686					
38	.18562	.06679	.04432	.03421	.02833	.02444	.02167	.01958	.01795	.01665	.01558	.01470	.01396	.01334
	.01281	.01236	.01198	.01170	.01140	.01120	.01100	.01090	.01080	.01070	.01070	.01080	.01090	.01110
	.01130	.01170	.01220	.01289	.01386	.01530	.01758	.02174	.03201	.17599				
39	.18388	.06622	.04395	.03394	.02810	.02424	.02148	.01941	.01779	.01649	.01543	.01455	.01381	.01319
	.01266	.01220	.01182	.01150	.01120	.01100	.01080	.01060	.01050	.01050	.01040	.01050	.01050	.01070
	.01080	.01110	.01150	.01199	.01268	.01365	.01509	.01736	.02150	.03171	.17516			
40	.18220	.06567	.04360	.03367	.02788	.02405	.02131	.01924	.01763	.01634	.01529	.01441	.01367	.01305
	.01251	.01206	.01170	.01130	.01100	.01080	.01060	.01040	.01030	.01020	.01020	.01020	.01020	.01030
	.01040	.01060	.01090	.01130	.01179	.01249	.01346	.01489	.01715	.02127	.03143	.17435		
41	.18057	.06513	.04326	.03340	.02766	.02386	.02114	.01909	.01749	.01620	.01515	.01427	.01354	.01291
	.01238	.01192	.01150	.01120	.01090	.01060	.01040	.01030	.01010	.01000	.00996	.00993	.00994	.00999
	.01010	.01020	.01040	.01070	.01110	.01160	.01230	.01327	.01470	.01695	.02105	.03115	.17357	
42	.17899	.06461	.04293	.03315	.02745	.02368	.02097	.01894	.01734	.01607	.01502	.01414	.01341	.01278
	.01225	.01178	.01140	.01100	.01070	.01050	.01030	.01010	.00995	.00984	.00975	.00971	.00969	.00971
	.00977	.00987	.01000	.01020	.01050	.01090	.01140	.01212	.01310	.01452	.01676	.02084	.03089	.17282
43	.17747	.06411	.04260	.03291	.02725	.02350	.02082	.01879	.01721	.01593	.01489	.01402	.01329	.01266
	.01212	.01170	.01130	.01090	.01060	.01030	.01010	.00994	.00979	.00966	.00957	.00950	.00947	.00946
	.00949	.00956	.00967	.00983	.01010	.01030	.01070	.01130	.01196	.01293	.01435	.01658	.02064	.03064
	.17209													
44	.17599	.06362	.04229	.03267	.02706	.02333	.02066	.01865	.01708	.01581	.01477	.01390	.01317	.01254
	.01201	.01150	.01110	.01080	.01050	.01020	.00999	.00980	.00963	.00950	.00939	.00931	.00926	.00924
	.00925	.00929	.00937	.00949	.00966	.00988	.01020	.01060	.01110	.01179	.01277	.01418	.01641	.02044
	.03039	.17138												
45	.17455	.06315	.04199	.03244	.02686	.02317	.02051	.01851	.01695	.01569	.01465	.01379	.01306	.01243
	.01189	.01140	.01100	.01070	.01040	.01010	.00986	.00966	.00949	.00935	.00923	.00914	.00908	.00904
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	.02026	.03016	.17070											
46	.17316	.06269	.04169	.03222	.02668	.02301	.02037	.01838	.01682	.01557	.01454	.01368	.01295	.01233
	.01179	.01130	.01090	.01060	.01020	.00998	.00974	.00954	.00936	.00921	.00909	.00899	.00891	.00886
	.00884	.00884	.00887	.00892	.00902	.00915	.00933	.00956	.00987	.01030	.01080	.01150	.01246	.01387
	.01608	.02008	.02993	.17003										
47	.17181	.06224	.04141	.03200	.02650	.02285	.02023	.01826	.01671	.01546	.01443	.01357	.01285	.01222
	.01170	.01120	.01080	.01050	.01010	.00987	.00963	.00942	.00924	.00908	.00895	.00884	.00876	.00870
	.00866	.00864	.00865	.00869	.00876	.00885	.00899	.00918	.00942	.00973	.01010	.01070	.01140	.01232
	.01372	.01592	.01990	.02971	.16939									
48	.17049	.06181	.04113	.03179	.02633	.02270	.02010	.01813	.01659	.01535	.01433	.01347	.01275	01213
	.01160	.01110	.01070	.01040	.01000	.00976	.00952	.00931	.00912	.00896	.00882	.00871	.00862	.00855
	.00850	.00847	.00846	.00848	.00852	.00860	.00870	.00884	.00903	.00928	.00959	.00999	.01050	01120
	.01219	.01358	.01577	.01974	.02950	.16876								
49	16921	.06139	.04086	.03158	.02616	.02255	.01997	.01801	.01648	.01524	.01423	.01337	.01265	01203
	.01150	.01100	01060	.01030	.00994	.00966	.00942	.00920	.00901	.00885	.00871	.00858	.00848	.00840
	.00835	.00831	.00829	.00829	.00831	.00837	.00844	.00855	.00870	.00890	.00914	.00946	.00986	.01040
	.01110	.01206	.01345	.01563	.01958	.02929	.16815							
50	16797	.06097	.04059	.03138	.02599	.02241	01984	01790	.01637	.01514	.01413	.01328	.01256	01194
	.01140	.01090	01050	.01020	.00985	.00957	.00932	.00910	.00891	.00874	.00859	.00847	.00836	.00827
	.00821	.00816	.00813	.00812	.00813	.00816	.00822	.00830	.00842	.00857	.00876	.00901	.00933	.00974
	.01030	.01100	.01193	.01332	.01549	.01942	.02909	.16756						

$K^* = .45000$
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1 1.00000

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4 .39867 .16285 .13444 .30403

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7 .31499 .12470 .08803 .07432 .07119 .08136 .24540

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9 .28406 .11260 .07812 .06358 .05654 .05404 .05623 .06863 .22623

10 .27207 .10800 .07464 .06022 .05273 .04904 .04836 .05157 .06432 .21908

11 .26166 .10400 .07172 .05753 .04987 .04562 .04372 .04411 .04792 .06082 .21300

12 .25252 .10060 .06923 .05530 .04761 .04307 .04057 .03973 .04078 .04497 .05791 .20774

13 .24439 .09753 .06706 .05340 .04574 .04107 .03823 .03677 .03661 .03810 .04253 .05544 .20312

14 .23710 .09481 .06514 .05175 .04416 .03943 .03641 .03459 .03381 .03410 .03589 .04048 .05331 .19901

15 .23050 .09234 .06342 .05030 .04280 .03805 .03492 .03290 .03175 .03142 .03204 .03403 .03872 .05146 .19534

16 .22450 .09010 .06186 .04901 .04161 .03687 .03368 .03153 .03016 .02946 .02946 .03030 .03244 .03719 .04982 .19202

17 .21899 .08805 .06045 .04784 .04055 .03584 .03262 .03039 .02887 .02795 .02758 .02780 .02881 .03106 .03584 .04837 .18899

18 .21393 .08615 .05916 .04678 .03959 .03492 .03170 .02941 .02781 .02673 .02613 .02599 .02639 .02753 .02985 .03465 .04706 .18623

19 .20924 .08440 .05796 .04581 .03873 .03410 .03088 .02857 .02690 .02573 .02497 .02460 .02464 .02517 .02640 .02878 .03358 .04587 .18368

20 .20488 .08277 .05685 .04491 .03793 .03335 .03015 .02782 .02612 .02487 .02401 .02348 .02329 .02347 .02410 .02541 .02783 .03262 .04480 .18132

21 .20082 .08126 .05582 .04408 .03721 .03268 .02949 .02716 .02543 .02413 .02320 .02256 .02221 .02216 .02244 .02316 .02452 .02697 .03175 .04381 .17913

22 .19702 .07983 .05486 .04331 .03653 .03205 .02889 .02656 .02481 .02349 .02250 .02179 .02133 .02112 .02117 .02154 .02232 .02372 .02620 .03096 .04291 .17709

23 .19345 .07849 .05396 .04259 .03591 .03148 .02834 .02602 .02426 .02291 .02188 .02112 .02058 .02026 .02016 .02030 .02074 .02157 .02300 .02549 .03023 .04207 .17517

24 .19009 .07723 .05311 .04192 .03532 .03095 .02783 .02552 .02376 .02240 .02134 .02054 .01994 .01954 .01933 .01932 .01953 .02002 .02089 .02235 .02484 .02956 .04130 .17338

25 .18692 .07604 .05231 .04128 .03477 .03045 .02736 .02507 .02330 .02193 .02086 .02002 .01938 .01892 .01863 .01851 .01856 .01883 .01936 .02027 .02175 .02425 .02894 .04058 .17169

26 .18392 .07492 .05155 .04068 .03426 .02998 .02693 .02464 .02288 .02151 .02042 .01956 .01889 .01839 .01803 .01783 .01777 .01789 .01820 .01877 .01971 .02120 .02370 .02837 .03990 .17009

27 .18108 .07385 .05083 .04012 .03377 .02954 .02652 .02425 .02250 .02112 .02002 .01914 .01845 .01791 .01751 .01725 .01711 .01712 .01728 .01763 .01823 .01919 .02070 .02319 .02783 .03927 .16858

28 .17839 .07283 .05015 .03958 .03331 .02913 .02613 .02388 .02214 .02076 .01965 .01876 .01805 .01749 .01705 .01674 .01655 .01647 .01653 .01673 .01711 .01774 .01871 .02023 .02272 .02733 .03868 .16715

29 .17582 .07186 .04950 .03907 .03288 .02874 .02577 .02354 .02180 .02042 .01932 .01842 .01769 .01711 .01665 .01630 .01606 .01592 .01589 .01599 .01622 .01664 .01728 .01827 .01979 .02228 .02687 .03812 .16579

30 .17337 .07093 .04888 .03858 .03247 .02837 .02543 .02321 .02149 .02011 .01900 .01810 .01736 .01676 .01628 .01590 .01563 .01544 .01536 .01537 .01550 .01576 .01620 .01686 .01786 .01939 .02187 .02643 .03760 .16449

31 .17104 .07005 .04829 .03811 .03207 .02802 .02511 .02291 .02119 .01982 .01871 .01780 .01706 .01645 .01595 .01555 .01524 .01502 .01489 .01484 .01489 .01505 .01534 .01579 .01647 .01748 .01901 .02148 .02601 .03710 .16325

32 .16881 .06920 .04772 .03767 .03169 .02769 .02480 .02262 .02091 .01955 .01844 .01753 .01678 .01616 .01564 .01523 .01490 .01465 .01448 .01439 .01437 .01445 .01464 .01495 .01542 .01611 .01712 .01865 .02112 .02562 .03663 .16207

33 .16667 .06839 .04718 .03725 .03134 .02737 .02451 .02234 .02065 .01929 .01819 .01727 .01652 .01589 .01537 .01493 .01459 .01431 .01411 .01398 .01393 .01395 .01405 .01425 .01458 .01507 .01577 .01679 .01832 .02078 .02525 .03618 .16094

34 .16462 .06761 .04666 .03684 .03099 .02706 .02423 .02208 .02040 .01905 .01795 .01703 .01627 .01564 .01511 .01467 .01430 .01401 .01379 .01363 .01353 .01351 .01355 .01368 .01390 .01424 .01474 .01545 .01648 .01801 .02045 .02490 .03575 .15986

35 .16265 .06687 .04615 .03645 .03066 .02677 .02396 .02183 .02016 .01882 .01772 .01681 .01605 .01541 .01487 .01442 .01404 .01374 .01349 .01331 .01319 .01312 .01312 .01319 .01333 .01357 .01393 .01444 .01515 .01618 .01771 .02015 .02457 .03535 .15882

36	.16076	.06615	.04567	.03607	.03034	.02649	.02371	.02159	.01993	.01860	.01750	.01660	.01583	.01519
	.01465	.01419	.01380	.01349	.01323	.01302	.01287	.01278	.01274	.01276	.01285	.01301	.01327	.01363
	.01415	.01487	.01591	.01743	.01986	.02425	.03496	.15782						
37	.15895	.06545	.04521	.03571	.03004	.02622	.02346	.02137	.01972	.01839	.01730	.01639	.01563	.01499
	.01444	.01398	.01358	.01325	.01298	.01276	.01259	.01248	.01241	.01239	.01243	.01254	.01271	.01298
	.01336	.01388	.01461	.01564	.01717	.01958	.02395	.03459	.15686					
38	.15720	.06478	.04476	.03536	.02975	.02597	.02323	.02115	.01951	.01819	.01711	.01620	.01544	.01480
	.01425	.01378	.01338	.01304	.01276	.01252	.01234	.01220	.01211	.01206	.01207	.01212	.01224	.01243
	.01271	.01310	.01363	.01436	.01539	.01691	.01932	.02367	.03424	.15593				
39	.15551	.06414	.04433	.03502	.02946	.02572	.02300	.02094	.01931	.01800	.01692	.01602	.01526	.01461
	.01406	.01359	.01318	.01284	.01255	.01230	.01211	.01195	.01184	.01177	.01174	.01177	.01184	.01197
	.01217	.01246	.01285	.01339	.01412	.01516	.01667	.01907	.02339	.03390	.15504			
40	.15388	.06352	.04391	.03470	.02919	.02548	.02279	.02074	.01913	.01782	.01675	.01585	.01509	.01444
	.01389	.01341	.01300	.01265	.01235	.01210	.01189	.01172	.01160	.01150	.01150	.01140	.01150	.01160
	.01170	.01193	.01222	.01262	.01316	.01390	.01493	.01645	.01883	.02313	.03357	.15418		
41	.15231	.06291	.04351	.03438	.02893	.02525	.02258	.02055	.01894	.01765	.01658	.01568	.01493	.01428
	.01373	.01325	.01283	.01248	.01217	.01191	.01170	.01150	.01140	.01130	.01120	.01120	.01120	.01120
	.01130	.01150	.01170	.01200	.01240	.01295	.01369	.01472	.01623	.01860	.02288	.03326	.15335	
42	.15079	.06233	.04312	.03408	.02867	.02503	.02238	.02036	.01877	.01748	.01642	.01553	.01477	.01413
	.01357	.01309	.01267	.01231	.01200	.01174	.01150	.01130	.01120	.01100	.01100	.01090	.01090	.01090
	.01100	.01110	.01120	.01150	.01178	.01219	.01274	.01349	.01452	.01602	.01838	.02264	.03296	.15255
43	.14932	.06176	.04274	.03379	.02843	.02481	.02218	.02018	.01860	.01732	.01626	.01538	.01462	.01398
	.01342	.01294	.01252	.01216	.01184	.01160	.01130	.01110	.01100	.01080	.01070	.01070	.01060	.01060
	.01070	.01070	.01090	.01100	.01130	.01160	.01200	.01255	.01329	.01433	.01582	.01818	.02241	.03267
	.15177													
44	.14789	.06122	.04237	.03350	.02819	.02460	.02200	.02001	.01844	.01717	.01611	.01523	.01448	.01384
	.01328	.01280	.01238	.01201	.01170	.01140	01120	.01100	.01080	.01070	.01050	.01050	.01040	01040
	.01040	.01040	.01050	.01070	.01080	.01110	.01140	.01181	.01237	.01311	.01414	.01563	.01798	.02218
	.03239	.15101												
45	.14651	.06069	.04202	.03322	.02796	.02440	.02181	.01984	.01828	.01702	.01597	.01509	.01434	.01370
	.01315	.01267	.01224	.01188	.01160	.01130	.01100	.01080	.01060	.01050	.01040	.01030	.01020	01020
	.01020	.01020	.01020	.01030	.01050	.01060	.01090	.01120	.01160	.01219	.01293	.01396	.01545	.01778
	.02197	.03212	.15028											
46	.14518	.06017	.04167	.03296	.02773	.02420	.02164	.01968	.01813	.01687	.01583	.01496	.01421	.01357
	.01302	.01254	.01211	.01174	.01140	.01110	.01090	.01070	.01050	.01030	.01020	.01010	.01000	.00996
	.00993	.00993	.00996	.01000	.01010	.01030	.01050	.01070	.01100	.01150	.01202	.01277	.01379	.01528
	.01760	.02177	.03186	.14958										
47	.14388	.05967	.04134	.03270	.02752	.02401	.02147	.01952	.01798	.01673	.01570	.01483	.01409	.01345
	.01290	.01241	.01199	.01160	.01130	.01100	.01080	.01050	.01030	.01020	.01000	.00993	.00984	.00977
	.00973	.00972	.00973	.00977	.00984	.00994	.01010	.01030	.01050	.01090	.01130	.01186	.01261	.01363
	.01511	.01742	.02157	.03161	.14889									
48	.14262	.05919	.04101	03244	.02731	.02383	.02130	.01937	.01784	.01660	.01557	.01470	.01397	.01333
	.01278	.01230	.01187	.01150	.01120	.01090	.01060	.01040	.01020	.01000	.00989	.00977	00968	00960
	.00955	.00952	.00951	.00953	.00958	.00966	.00977	.00992	.01010	.01040	.01070	.01110	.01170	.01245
	.01347	.01495	.01725	.02138	.03137	.14822								
49	.14140	.05871	.04070	.03220	.02710	.02365	.02114	.01922	01770	.01647	.01545	.01458	.01385	.01322
	.01266	.01218	.01176	.01140	.01110	.01080	.01050	.01030	.01010	.00991	.00976	.00963	.00952	.00944
	.00938	.00934	.00932	.00932	.00935	.00940	.00948	.00960	.00976	.00997	.01020	01060	.01100	.01160
	.01230	.01332	.01479	.01708	.02119	.03114	.14757							
50	.14021	05825	.04039	.03196	.02690	.02348	.02099	.01908	.01757	01634	.01533	.01447	01374	.01310
	.01256	.01207	.01170	.01130	.01090	.01070	.01040	.01020	.00996	.00978	.00963	.00950	00938	.00929
	.00922	.00917	.00914	.00912	.00914	.00917	.00923	.00932	.00944	.00961	.00982	.01010	.01040	.01090
	.01140	.01216	.01318	.01464	.01692	.02101	.03091	.14694						

K* = .50000
N = 50

n	
1	1.00000
2	.54564 .45436
3	.43818 .21492 .34690
4	.37989 .17489 .14749 .29774
5	.34135 .15414 .11860 .11760 .26833
6	.31325 .14049 .10440 .09322 .10040 .24825
7	.29151 .13047 .09530 .08167 .07857 .08906 .23340
8	.27399 .12260 .08874 .07446 .06847 .06892 .08093 .22184
9	.25946 .11630 .08366 .06933 .06226 .05974 .06203 .07477 .21250
10	.24713 .11090 .07953 .06540 .05790 .05417 .05350 .05683 .06991 .20473
11	.23650 .10630 .07608 .06223 .05459 .05029 .04837 .04878 .05274 .06596 .19813
12	.22720 .10230 .07312 .05959 .05195 .04738 .04483 .04398 .04509 .04943 .06267 .19243
13	.21897 .09880 .07055 .05734 .04976 .04506 .04218 .04069 .04054 .04210 .04669 .05988 .18745
14	.21161 .09565 .06827 .05538 .04790 .04316 .04009 .03824 .03744 .03776 .03962 .04437 .05747 .18302
15	.20498 .09282 .06623 .05365 .04629 .04154 .03838 .03632 .03515 .03481 .03546 .03754 .04238 .05537 .17907
16	.19896 .09025 .06440 .05211 .04487 .04015 .03694 .03475 .03335 .03264 .03264 .03352 .03575 .04065 .05352 .17550
17	.19347 .08791 .06273 .05072 .04361 .03893 .03570 .03344 .03189 .03094 .03056 .03080 .03185 .03419 .03913 .05187 .17226
18	.18842 .08575 .06120 .04946 .04247 .03784 .03461 .03231 .03067 .02957 .02895 .02881 .02923 .03041 .03283 .03778 .05039 .16929
19	.18377 .08376 .05979 .04830 .04143 .03687 .03365 .03132 .02962 .02842 .02764 .02726 .02731 .02787 .02915 .03163 .03658 .04905 .16657
20	.17946 .08191 .05849 .04723 .04049 .03598 .03279 .03045 .02871 .02744 .02656 .02602 .02582 .02601 .02668 .02803 .03055 .03549 .04783 .16405
21	.17545 .08020 .05728 .04625 .03962 .03517 .03201 .02967 .02791 .02659 .02563 .02498 .02463 .02457 .02487 .02562 .02703 .02958 .03450 .04672 .16172
22	.17171 .07859 .05615 .04533 .03881 .03443 .03129 .02896 .02719 .02585 .02483 .02410 .02363 .02342 .02348 .02386 .02468 .02614 .02870 .03360 .04570 .15954
23	.16820 .07709 .05510 .04447 .03806 .03374 .03064 .02832 .02655 .02518 .02413 .02334 .02279 .02246 .02236 .02251 .02296 .02383 .02532 .02790 .03278 .04475 .15751
24	.16491 .07567 .05411 .04367 .03737 .03311 .03003 .02773 .02596 .02458 .02350 .02268 .02207 .02165 .02144 .02142 .02164 .02216 .02307 .02459 .02717 .03202 .04387 .15560
25	.16182 .07434 .05317 .04292 .03671 .03251 .02947 .02719 .02542 .02403 .02294 .02208 .02143 .02096 .02066 .02053 .02059 .02087 .02143 .02237 .02391 .02650 .03132 .04306 .15380
26	.15889 .07308 .05229 .04221 .03610 .03195 .02895 .02668 .02492 .02353 .02243 .02155 .02086 .02035 .01998 .01977 .01972 .01984 .02017 .02076 .02173 .02329 .02587 .03066 .04230 .15211
27	.15613 .07188 .05146 .04154 .03552 .03143 .02846 .02621 .02446 .02307 .02196 .02107 .02036 .01980 .01940 .01912 .01898 .01899 .01916 .01953 .02016 .02115 .02272 .02530 .03006 .04158 .15051
28	.15351 .07075 .05067 .04090 .03497 .03093 .02800 .02577 .02403 .02265 .02153 .02063 .01990 .01932 .01887 .01855 .01835 .01827 .01833 .01854 .01895 .01960 .02061 .02219 .02476 .02949 .04091 .14900
29	.15102 .06967 .04991 .04030 .03445 .03047 .02757 .02536 .02363 .02225 .02113 .02022 .01948 .01888 .01841 .01805 .01780 .01766 .01763 .01773 .01798 .01841 .01909 .02011 .02169 .02426 .02896 .04028 .14756
30	.14865 .06865 .04920 .03972 .03396 .03002 .02716 .02497 .02326 .02188 .02077 .01985 .01910 .01848 .01799 .01760 .01732 .01713 .01704 .01705 .01719 .01746 .01792 .01861 .01965 .02124 .02380 .02846 .03969 .14619
31	.14640 .06767 .04851 .03918 .03349 .02960 .02677 .02461 .02291 .02154 .02042 .01950 .01874 .01812 .01761 .01720 .01688 .01665 .01652 .01647 .01652 .01669 .01699 .01746 .01817 .01922 .02081 .02336 .02799 .03913 .14489
32	.14424 .06673 .04786 .03865 .03304 .02920 .02640 .02426 .02257 .02121 .02010 .01918 .01842 .01778 .01726 .01683 .01649 .01623 .01606 .01596 .01595 .01603 .01622 .01655 .01704 .01776 .01882 .02040 .02294 .02755 .03860 .14364
33	.14219 .06584 .04724 .03815 .03261 .02882 .02605 .02393 .02226 .02091 .01980 .01888 .01811 .01747 .01693 .01649 .01614 .01585 .01565 .01551 .01545 .01547 .01558 .01580 .01614 .01665 .01738 .01844 .02003 .02256 .02713 .03809 .14245
34	.14022 .06498 .04664 .03768 .03220 .02846 .02572 .02362 .02196 .02062 .01951 .01859 .01783 .01718 .01664 .01618 .01581 .01551 .01528 .01511 .01502 .01499 .01504 .01517 .01540 .01576 .01628 .01702 .01809 .01967 .02219 .02673 .03761 .14131
35	.13833 .06416 .04606 .03722 .03181 .02611 .02540 .02332 .02167 .02034 .01924 .01833 .01756 .01691 .01636 .01590 .01551 .01520 .01495 .01476 .01463 .01456 .01456 .01463 .01478 .01503 .01541 .01594 .01668 .01775 .01933 .02184 .02636 .03715 .14022

36	.13652	.06337	.04551	.03678	.03144	.02778	.02509	.02303	.02140	.02008	.01899	.01807	.01730	.01665
	.01610	.01563	.01524	.01491	.01464	.01443	.01428	.01418	.01414	.01416	.01425	.01442	.01469	.01507
	.01561	.01637	.01744	.01902	.02151	.02600	.03672	.13917						
37	.13478	.06261	.04498	.03635	.03108	.02746	.02480	.02276	.02114	.01983	.01874	.01783	.01707	.01641
	.01586	.01538	.01498	.01464	.01436	.01414	.01396	.01384	.01377	.01375	.01380	.01390	.01409	.01437
	.01476	.01531	.01607	.01714	.01872	.02120	.02566	.03630	.13816					
38	.13311	.06188	04447	.03595	.03073	.02715	.02452	.02250	.02090	.01959	.01851	.01761	.01684	.01619
	.01563	.01515	.01474	.01439	.01410	.01386	.01367	.01353	.01343	.01339	.01339	.01345	.01358	.01378
	.01407	.01447	.01502	.01579	.01686	.01843	.02090	.02533	.03590	.13719				
39	.13150	.06117	.04398	.03555	.03039	.02685	.02425	.02225	.02066	.01936	.01829	.01739	.01663	.01597
	.01541	.01493	.01452	.01416	.01386	.01361	.01341	.01325	.01313	.01306	.01303	.01306	.01313	.01327
	.01348	.01378	.01419	.01475	.01552	.01659	.01816	.02062	.02502	.03552	.13626			
40	.12995	.06049	.04350	.03518	.03007	.02657	.02399	.02201	.02043	.01915	.01808	.01718	01642	.01577
	.01521	.01473	.01431	.01395	.01364	.01338	.01316	.01299	.01286	.01277	.01271	.01271	.01274	.01284
	.01299	.01321	.01351	.01393	.01450	.01526	.01634	.01790	.02035	.02472	.03515	.13535		
41	.12845	.05984	.04304	.03481	.02976	.02629	.02374	.02178	.02021	.01894	.01788	.01699	.01623	.01558
	.01502	.01453	.01411	.01375	.01343	.01316	.01294	.01275	.01261	.01250	.01242	.01239	.01240	.01245
	.01255	.01272	.01295	.01326	.01368	.01425	.01502	.01610	.01765	.02009	.02444	.03480	.13448	
42	.12701	.05920	.04260	.03446	.02946	.02603	.02350	.02155	.02000	.01874	.01769	.01680	.01604	.01539
	.01483	.01435	.01392	.01355	.01324	.01296	.01273	.01253	.01237	.01225	.01216	.01211	.01209	.01211
	.01218	.01229	.01246	.01270	.01302	.01345	.01402	.01479	.01587	.01742	.01984	.02417	.03446	.13364
43	.12561	.05859	.04217	.03412	.02917	.02577	.02327	.02134	.01980	.01855	.01750	.01662	.01587	.01522
	.01466	.01417	.01375	.01337	.01305	.01277	.01253	.01233	.01216	.01202	.01192	.01185	.01181	.01181
	.01184	.01192	.01204	.01222	.01247	.01280	.01323	.01380	.01458	.01565	.01719	.01960	.02390	.03414
	.13283													
44	.12430	.05800	.04176	.03379	.02889	.02553	.02305	.02113	.01961	.01836	.01732	.01645	.01570	.01505
	.01449	.01400	.01358	.01320	.01288	.01259	.01235	.01214	.01196	.01181	.01170	.01160	.01160	01150
	.01150	.01160	.01170	.01181	.01200	.01225	.01258	.01302	.01359	.01437	.01544	.01698	.01938	.02365
	.03382	.13204												
45	.12300	.05742	.04135	.03347	.02862	.02529	.02283	.02093	.01942	.01818	.01715	.01628	.01553	.01489
	.01433	.01384	.01342	.01304	.01271	.01243	.01217	.01196	.01177	.01160	.01150	.01140	01130	.01130
	.01130	.01130	.01140	.01140	.01160	.01178	.01204	.01238	.01282	.01340	.01417	.01523	.01677	.01916
	.02341	.03352	.13128											
46	.12170	.05687	.04097	.03315	.02836	.02505	02262	.02074	.01924	.01801	.01698	.01612	.01538	01474
	.01418	.01369	.01326	.01289	.01256	.01227	.01201	.01179	.01160	.01140	.01130	.01120	.01110	.01110
	.01100	.01100	.01110	.01110	.01120	.01140	.01160	.01184	.01218	.01262	.01321	.01398	.01504	.01657
	.01895	.02318	.03323	.13054										
47	.12050	.05633	.04059	.03285	.02810	.02483	.02241	.02055	.01906	.01784	.01682	01596	.01523	.01459
	01403	.01355	.01312	.01274	.01241	.01211	.01186	.01160	.01140	.01130	.01110	.01100	.01090	.01080
	.01080	.01080	.01080	.01080	.01090	.01100	.01120	.01140	.01160	.01200	.01244	.01302	01380	.01486
	.01638	.01875	.02295	.03295	.12983									
48	.11930	.05580	.04022	.03256	.02785	.02461	.02222	.02037	.01889	.01768	.01667	.01581	.01508	01445
	.01389	.01341	.01298	.01260	.01227	.01197	.01170	.01150	.01130	.01110	.01100	.01080	.01070	.01070
	.01060	.01060	.01060	.01060	.01060	.01070	.01080	.01100	.01120	.01150	.01182	.01227	.01285	.01362
	.01468	.01620	.01855	.02273	.03267	.12913								
49	.11810	05529	.03986	.03228	.02761	.02440	02203	.02019	01873	.01752	.01652	.01567	.01494	.01431
	.01376	.01327	.01285	.01247	.01213	.01183	.01160	.01130	.01110	.01100	.01080	.01070	.01060	.01050
	01040	.01040	.01030	.01030	.01040	.01040	.01050	.01060	.01080	.01100	.01130	.01160	.01210	.01268
	01345	.01451	.01602	.01836	.02253	.03241	.12846							
50	.11700	.05480	.03952	.03200	.02738	.02419	.02184	.02002	.01857	.01737	.01638	.01553	.01481	01418
	.01363	.01314	.01272	.01234	.01200	.01170	.01140	.01120	.01100	.01080	.01070	.01050	.01040	.01030
	.01020	.01020	.01010	.01010	.01010	.01020	.01020	.01030	.01050	.01060	.01090	.01110	.01150	.01194
	.01252	.01329	.01435	.01585	.01818	.02232	.03216	.12780						

$K^* = .55000$
$N = 50$

n

1 1.00000

2 .53966 .46034

3 .42436 .23061 .34503

4 .36240 .18587 .16004 .29169

5 .32185 .16221 .12843 .12777 .25974

6 .29254 .14655 .11240 .10160 .10890 .23796

7 .27003 .13504 .10200 .08881 .08581 .09637 .22190

8 .25202 .12605 .09447 .08065 .07476 .07528 .08734 .20942

9 .23718 .11880 .08857 .07478 .06783 .06535 .06770 .08047 .19937

10 .22466 .11270 .08378 .07024 .06290 .05920 .05855 .06194 .07504 .19103

11 .21392 .10740 .07976 .06656 .05911 .05486 .05295 .05339 .05741 .07062 .18397

12 .20456 .10290 .07632 .06349 .05607 .05156 .04902 .04819 .04932 .05373 .06694 .17788

13 .19632 .09895 .07333 .06086 .05353 .04891 .04605 .04457 .04443 .04602 .05067 .06382 .17256

14 .18898 .09541 .07069 .05857 .05136 .04672 .04368 .04184 .04104 .04137 .04327 .04808 .06112 .16786

15 .18239 .09224 .06833 .05655 .04946 .04485 .04173 .03968 .03850 .03817 .03884 .04095 .04585 .05877 .16366

16 .17643 .08937 .06621 .05475 .04782 .04324 .04007 .03790 .03650 .03578 .03579 .03669 .03896 .04391 .05669 .15988

17 .17101 .08675 .06429 .05313 .04634 .04181 .03864 .03640 .03485 .03390 .03352 .03377 .03485 .03723 .04221 .05484 .15646

18 .16605 .08436 .06253 .05165 .04501 .04054 .03738 .03510 .03347 .03237 .03174 .03161 .03204 .03325 .03571 .04070 .05318 .15333

19 .16148 .08215 .06091 .05030 .04380 .03940 .03626 .03396 .03227 .03107 .03029 .02991 .02996 .03054 .03184 .03436 .03934 .05168 .15046

20 .15727 .08012 .05942 .04906 .04269 .03836 .03525 .03295 .03123 .02996 .02908 .02853 .02834 .02853 .02921 .03059 .03315 .03812 .05032 .14781

21 .15335 .07822 .05804 .04791 .04167 .03742 .03434 .03204 .03031 .02899 .02803 .02738 .02702 .02697 .02727 .02804 .02948 .03206 .03701 .04907 .14535

22 .14971 .07646 .05675 .04685 .04073 .03654 .03351 .03122 .02948 .02814 .02712 .02639 .02592 .02570 .02577 .02616 .02699 .02848 .03108 .03600 .04793 .14307

23 .14631 .07481 .05555 .04585 .03985 .03574 .03274 .03047 .02873 .02737 .02632 .02553 .02498 .02465 .02455 .02470 .02516 .02605 .02757 .03018 .03508 .04687 .14094

24 .14313 .07326 .05442 .04492 .03904 .03499 .03203 .02978 .02804 .02668 .02560 .02478 .02416 .02375 .02353 .02352 .02375 .02427 .02520 .02674 .02936 .03422 .04589 .13895

25 .14014 .07181 .05336 .04405 .03827 .03429 .03137 .02915 .02741 .02604 .02496 .02410 .02344 .02297 .02267 .02254 .02260 .02289 .02346 .02442 .02599 .02860 .03343 .04497 .13707

26 .13732 .07044 .05236 .04323 .03755 .03363 .03076 .02856 .02683 .02546 .02437 .02349 .02280 .02228 .02192 .02170 .02165 .02177 .02211 .02272 .02371 .02529 .02790 .03270 .04412 .13531

27 .13466 .06914 .05142 .04246 .03688 .03302 .03018 .02801 .02630 .02493 .02383 .02294 .02223 .02167 .02126 .02098 .02084 .02085 .02103 .02140 .02204 .02306 .02465 .02726 .03202 .04333 .13364

28 .13214 .06791 .05053 .04172 .03624 .03244 .02964 .02749 .02579 .02443 .02333 .02243 .02170 .02112 .02067 .02035 .02015 .02007 .02013 .02034 .02076 .02142 .02246 .02405 .02665 .03138 .04258 .13206

29 .12976 .06675 .04968 .04103 .03563 .03189 .02913 .02701 .02533 .02397 .02287 .02196 .02122 .02062 .02015 .01979 .01953 .01939 .01937 .01947 .01972 .02016 .02085 .02190 .02350 .02609 .03079 .04187 .13057

30 .12750 .06564 .04887 .04037 .03506 .03137 .02865 .02655 .02488 .02354 .02244 .02153 .02078 .02017 .01967 .01928 .01899 .01880 .01871 .01873 .01887 .01915 .01961 .02032 .02138 .02299 .02557 .03023 .04121 .12915

31 .12534 .06459 .04811 .03974 .03451 .03088 .02819 .02612 .02447 .02313 .02204 .02113 .02037 .01975 .01924 .01882 .01850 .01828 .01814 .01809 .01814 .01831 .01862 .01911 .01983 .02090 .02250 .02507 .02970 .04058 .12779

32 .12330 .06358 .04737 .03914 .03399 .03041 .02776 .02571 .02408 .02275 .02166 .02075 .01999 .01936 .01884 .01841 .01806 .01780 .01763 .01753 .01752 .01761 .01780 .01813 .01863 .01937 .02044 .02205 .02461 .02920 .03999 .12650

33 .12130 .06262 .04668 .03857 .03350 .02997 .02735 .02532 .02371 .02239 .02131 .02040 .01964 .01900 .01847 .01802 .01766 .01738 .01717 .01703 .01697 .01700 .01711 .01733 .01768 .01820 .01894 .02002 .02163 .02417 .02873 .03943 .12527

34 .11950 .06170 .04601 .03802 .03303 .02954 .02696 .02495 .02336 .02205 .02097 .02007 .01931 .01866 .01812 .01767 .01729 .01699 .01676 .01659 .01649 .01647 .01651 .01665 .01689 .01726 .01779 .01854 .01962 .02123 .02376 .02828 .03889 .12410

35 .11770 .06082 .04537 .03750 .03257 .02914 .02658 .02460 .02302 .02173 .02065 .01976 .01899 .01835 .01780 .01734 .01695 .01664 .01638 .01619 .01606 .01599 .01599 .01607 .01622 .01648 .01686 .01740 .01816 .01925 .02085 .02337 .02786 .03838 .12300

36	.11600	.05998	.04475	.03700	.03214	.02875	.02623	.02427	.02270	.02142	.02035	.01946	.01870	.01805
	.01750	.01704	.01664	.01631	.01604	.01583	.01567	.01557	.01554	.01556	.01565	.01583	.01610	.01649
	.01704	.01781	.01889	.02049	.02300	.02746	.03789	.12190						
37	.11430	.05917	.04416	.03652	.03172	.02838	.02588	.02395	.02240	.02113	.02007	.01918	.01842	.01778
	.01722	.01675	.01635	.01601	.01572	.01550	.01532	.01520	.01513	.01511	.01515	.01527	.01545	.01574
	.01614	.01670	.01747	.01856	.02015	.02265	.02708	.03743	.12080					
38	.11270	.05839	.04360	.03606	.03132	.02802	.02556	.02364	.02211	.02085	.01980	.01891	.01816	.01751
	.01696	.01648	.01607	.01572	.01543	.01519	.01500	.01485	.01476	.01471	.01471	.01478	.01490	.01511
	.01540	.01581	.01638	.01716	.01824	.01983	.02231	.02671	.03698	.11980				
39	.11120	.05764	.04305	.03561	.03094	.02767	.02524	.02335	.02183	.02058	.01954	.01866	.01791	.01726
	.01671	.01623	.01581	.01546	.01516	.01491	.01470	.01454	.01442	.01435	.01432	.01434	.01442	.01457
	.01478	.01509	.01551	.01608	.01686	.01794	.01953	.02199	.02636	.03656	.11890			
40	.10980	.05692	.04253	.03518	.03057	.02734	.02494	.02306	.02156	.02032	.01929	.01842	.01767	.01703
	.01647	.01599	.01557	.01521	.01490	.01464	.01442	.01425	.01411	.01402	.01397	.01396	.01400	.01409
	.01425	.01447	.01479	.01521	.01579	.01657	.01766	.01923	.02169	.02603	.03615	.11800		
41	.10840	.05623	.04202	.03477	.03021	.02702	.02464	.02279	.02130	.02008	.01905	.01818	.01744	.01680
	.01625	.01576	.01534	.01498	.01466	.01440	.01417	.01398	.01383	.01372	.01365	.01361	.01362	.01368
	.01378	.01395	.01418	.01451	.01494	.01552	.01630	.01739	.01896	.02140	.02571	.03576	.11710	
42	.10700	.05556	.04153	.03437	.02987	.02672	.02436	.02253	.02105	.01984	.01882	.01796	.01722	.01659
	.01603	.01555	.01513	.01476	.01444	.01416	.01393	.01373	.01357	.01345	.01336	.01330	.01328	.01331
	.01337	.01349	.01367	.01391	.01424	.01468	.01526	.01604	.01713	.01869	.02112	.02540	.03538	.11620
43	.10570	.05491	.04106	.03398	.02953	.02642	.02409	.02228	.02082	.01961	.01861	.01775	.01702	.01638
	.01583	.01534	.01492	.01455	.01423	.01395	.01371	.01350	.01333	.01319	.01309	.01301	.01298	.01297
	.01301	.01309	.01322	.01340	.01365	.01399	.01443	.01501	.01580	.01688	.01844	.02085	.02511	.03502
	.11540													
44	.10440	.05428	.04060	.03361	.02921	.02613	.02383	.02203	.02059	.01940	.01840	.01755	.01682	.01618
	.01563	.01515	.01473	.01435	.01403	.01374	.01350	.01328	.01310	.01296	.01284	.01275	.01270	.01267
	.01268	.01273	.01282	.01296	.01315	.01340	.01374	.01419	.01478	.01556	.01664	.01820	.02060	.02483
	.03467	.11460												
45	.10320	.05368	.04016	.03325	.02890	.02586	.02358	.02180	.02037	.01918	.01819	.01735	.01662	.01599
	.01544	.01496	.01454	.01417	.01384	.01355	.01330	.01308	.01289	.01274	01261	.01251	.01244	.01240
	.01239	.01241	.01247	.01257	.01271	.01291	.01317	.01352	.01397	.01455	.01534	.01642	.01796	02035
	.02456	.03434	.11380											
46	.10200	.05309	.03973	.03290	.02860	.02559	.02333	.02157	.02015	.01898	.01800	.01716	.01644	.01581
	.01527	.01478	.01436	.01399	.01366	.01337	.01311	.01289	.01269	.01253	.01239	.01228	.01220	.01214
	.01212	.01212	.01215	.01222	.01232	.01248	.01268	.01295	.01330	.01375	.01434	.01513	.01620	.01774
	.02012	.02429	.03401	.11300										
47	.10090	.05252	.03932	.03256	.02831	.02533	.02310	.02135	.01995	.01879	.01781	.01698	.01626	.01564
	.01509	.01461	.01419	.01382	.01348	.01319	.01293	.01271	.01251	.01234	.01219	.01207	.01198	.01191
	.01187	.01185	.01186	.01191	.01198	.01210	.01225	.01246	.01274	01309	.01354	.01414	.01492	.01599
	.01753	.01989	.02404	.03370	.11230									
48	.09979	.05197	.03892	.03224	.02803	.02508	.02287	.02114	.01975	.01860	.01763	.01680	.01609	01547
	.01493	.01445	.01403	.01365	.01332	.01303	.01276	.01253	.01233	.01216	.01201	.01188	.01178	.01170
	.01160	.01160	.01160	.01160	.01170	.01176	.01188	.01204	.01226	.01254	.01289	.01335	.01394	.01472
	.01579	.01732	.01967	.02380	.03340	.11160								
49	.09871	.05144	.03853	.03192	.02775	.02483	.02265	.02093	.01955	.01841	.01745	.01663	.01593	01531
	.01477	.01429	.01387	.01350	.01316	.01287	.01260	.01237	.01216	.01198	.01183	.01170	.01160	.01150
	.01140	.01140	.01140	.01140	.01140	.01150	.01150	.01170	.01184	.01206	.01234	.01270	.01316	.01375
	.01454	.01560	.01712	.01946	.02357	.03311	.11090							
50	.09768	.05092	.03815	.03161	.02749	.02460	.02243	.02073	.01937	.01823	.01728	.01647	.01577	.01515
	.01462	01414	.01372	.01335	.01301	.01272	.01245	.01222	.01201	.01182	.01170	.01150	.01140	.01130
	.01120	01120	.01110	.01110	.01110	.01120	.01120	.01130	.01150	.01170	.01187	01216	.01252	.01298
	.01357	.01435	.01542	.01693	.01926	.02334	.03282	.11020						

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9 .21702 .12020 .09290 .07992 .07325 .07086 .07322 .08571 .18687

10 .20442 .11340 .08742 .07474 .06772 .06414 .06353 .06690 .07970 .17802

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13 .17615 .09819 .07548 .06397 .05705 .05261 .04984 .04839 .04827 .04985 .05445 .06723 .15851

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18 .14643 .08222 .06324 .05339 .04724 .04302 .03999 .03779 .03620 .03512 .03451 .03438 .03482 .03602 .03846 .04338 .05542 .13837

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20 .13791 .07762 .05974 .05043 .04456 .04050 .03755 .03533 .03367 .03243 .03156 .03103 .03084 .03103 .03171 .03309 .03562 .04051 .05224 .13264

21 .13413 .07558 .05820 .04912 .04339 .03941 .03649 .03429 .03261 .03133 .03039 .02975 .02940 .02935 .02966 .03042 .03186 .03441 .03927 .05086 .13009

22 .13062 .07369 .05676 .04791 .04230 .03840 .03553 .03334 .03166 .03036 .02937 .02865 .02819 .02797 .02804 .02843 .02927 .03074 .03332 .03815 .04959 .12773

23 .12735 .07192 .05542 .04678 .04130 .03747 .03464 .03248 .03080 .02948 .02846 .02769 .02714 .02682 .02672 .02687 .02734 .02822 .02973 .03232 .03711 .04841 .12553

24 .12430 .07026 .05416 .04572 .04036 .03660 .03382 .03168 .03001 .02869 .02765 .02684 .02623 .02583 .02561 .02560 .02583 .02635 .02728 .02882 .03140 .03616 .04733 .12350

25 .12140 .06871 .05299 .04473 .03948 .03579 .03306 .03095 .02929 .02796 .02691 .02607 .02543 .02496 .02466 .02454 .02460 .02489 .02546 .02642 .02797 .03056 .03528 .04632 .12150

26 .11870 .06725 .05188 .04380 .03866 .03504 .03235 .03026 .02862 .02730 .02623 .02538 .02471 .02419 .02383 .02362 .02357 .02370 .02403 .02464 .02563 .02720 .02978 .03447 .04537 .11970

27 .11620 .06587 .05084 .04293 .03788 .03433 .03168 .02963 .02800 .02668 .02561 .02475 .02405 .02351 .02310 .02283 .02269 .02270 .02288 .02326 .02390 .02491 .02649 .02906 .03371 .04449 .11800

28 .11380 .06457 .04985 .04210 .03715 .03366 .03106 .02903 .02742 .02611 .02504 .02417 .02346 .02289 .02245 .02213 .02193 .02185 .02191 .02213 .02254 .02321 .02424 .02582 .02839 .03300 .04366 .11640

29 .11160 .06334 .04892 .04132 .03646 .03303 .03047 .02847 .02687 .02558 .02451 .02363 .02291 .02233 .02186 .02150 .02126 .02112 .02109 .02119 .02145 .02189 .02257 .02362 .02521 .02776 .03233 .04289 .11490

30 .10940 .06217 .04804 .04058 .03581 .03244 .02991 .02794 .02636 .02508 .02402 .02314 .02241 .02181 .02132 .02094 .02065 .02046 .02038 .02039 .02053 .02082 .02128 .02199 .02304 .02463 .02717 .03171 .04215 .11340

31 .10740 .06106 .04719 .03988 .03519 .03187 .02938 .02744 .02588 .02461 .02355 .02268 .02194 .02133 .02083 .02043 .02011 .01988 .01975 .01970 .01976 .01993 .02024 .02072 .02144 .02250 .02409 .02662 .03112 .04146 .11200

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33 .10360 .05899 .04563 .03857 .03404 .03083 .02841 .02652 .02500 .02375 .02271 .02184 .02110 .02048 .01996 .01952 .01917 .01889 .01868 .01855 .01849 .01851 .01863 .01884 .01919 .01971 .02045 .02153 .02312 .02562 .03004 .04018 .10950

34 .10180 .05803 .04490 .03796 .03350 .03034 .02796 .02609 .02459 .02335 .02232 .02145 .02072 .02009 .01957 .01912 .01875 .01846 .01822 .01806 .01796 .01794 .01798 .01812 .01836 .01873 .01926 .02001 .02108 .02267 .02516 .02954 .03959 .10830

35 .10020 .05711 .04420 .03737 .03299 .02988 .02753 .02569 .02420 .02298 .02196 .02109 .02036 .01973 .01920 .01875 .01837 .01806 .01781 .01762 .01749 .01742 .01742 .01750 .01765 .01791 .01829 .01883 .01959 .02067 .02225 .02472 .02907 .03903 .10710

36	.09855	.05623	.04353	.03682	.03250	.02943	.02712	.02530	.02383	.02262	.02161	.02075	.02002	.01939
	.01886	.01840	.01801	.01769	.01742	.01721	.01706	.01696	01692	.01695	.01704	.01722	.01749	.01788
	.01843	.01919	.02027	.02185	.02431	.02863	.03849	.10600						
37	.09702	.05539	.04289	.03628	.03203	.02901	.02672	.02493	.02348	.02228	.02128	.02043	01970	.01907
	.01854	.01807	.01768	.01734	.01707	.01684	.01667	.01655	.01648	.01646	.01651	.01662	.01681	01709
	.01749	.01805	.01882	.01990	.02147	.02391	.02820	.03798	.10500					
38	.09554	.05457	.04228	.03577	.03158	.02860	.02635	.02458	.02314	.02196	.02096	.02012	.01939	.01877
	.01823	.01777	.01737	.01702	.01674	.01650	.01631	.01617	.01607	.01602	.01603	.01609	.01622	.01642
	.01672	.01713	.01769	.01847	.01955	.02111	.02354	.02779	.03749	.10390				
39	.09412	.05380	.04169	.03527	.03114	.02821	.02599	.02424	.02282	.02165	.02066	.01982	.01910	.01848
	.01794	.01748	.01707	.01673	.01643	.01618	.01598	.01582	.01570	.01563	.01560	.01563	.01571	.01585
	.01606	.01637	.01679	.01736	.01813	.01921	.02077	.02318	.02741	.03703	.10300			
40	.09276	.05305	.04112	.03480	.03073	.02783	.02564	.02391	.02251	.02135	.02037	.01954	.01883	.01821
	.01767	.01720	.01680	.01644	.01614	.01588	.01567	.01549	.01536	.01527	.01522	.01521	.01525	.01534
	.01550	.01572	.01604	.01646	.01704	.01781	.01889	.02044	.02284	.02704	.03658	.10200		
41	.09146	.05233	.04057	.03434	03033	.02747	.02530	.02360	.02221	.02107	02010	.01927	.01856	.01795
	.01741	.01694	.01653	.01618	.01587	.01560	.01538	.01520	.01505	.01494	.01487	.01483	.01484	.01490
	.01500	.01517	.01540	.01573	.01616	.01673	.01751	.01859	.02013	.02252	.02668	.03615	.10110	
42	.09020	.05163	.04005	.03390	.02994	.02712	.02498	.02330	.02193	.02079	.01983	.01902	.01831	.01770
	.01716	.01669	.01628	.01593	.01561	.01534	.01511	.01492	.01476	.01463	.01454	.01449	.01447	01450
	.01457	.01468	.01486	.01510	.01543	.01587	.01645	.01723	.01830	.01984	.02221	.02634	.03573	.10030
43	.08898	.05096	.03954	.03348	.02957	.02678	.02467	.02301	.02165	.02053	.01958	.01877	.01807	.01746
	.01693	.01646	.01605	.01569	.01537	.01510	.01486	.01466	.01449	.01435	.01425	.01418	.01414	.01414
	.01417	.01425	.01438	.01456	.01481	.01515	.01559	.01617	.01695	.01802	.01956	.02191	.02601	.03534
	.09942													
44	.08781	.05032	.03905	.03307	.02921	.02646	.02437	.02273	.02139	.02028	.01934	.01853	.01784	.01723
	.01670	.01623	.01582	.01546	.01514	.01486	.01462	.01441	.01424	.01409	.01397	.01389	.01383	01381
	.01382	.01387	.01396	.01409	.01428	.01454	.01488	.01533	.01591	.01669	.01776	.01928	.02163	.02570
	.03495	.09860												
45	.08668	.04969	.03857	.03267	.02886	.02614	.02409	.02246	.02113	.02003	.01910	.01830	.01761	.01701
	.01648	.01602	.01561	.01525	.01493	.01464	.01440	.01418	.01400	.01384	.01372	.01362	.01355	.01351
	.01350	.01352	.01358	.01368	.01382	.01402	.01428	.01463	.01508	.01566	.01644	.01751	.01902	.02135
	.02540	.03458	.09782											
46	.08559	.04909	.03811	.03229	.02853	.02584	.02381	.02220	.02089	.01980	.01888	.01809	.01740	01680
	.01628	.01581	.01540	.01504	.01472	.01443	.01418	.01396	.01377	.01361	.01348	01337	.01329	.01323
	.01320	.01320	.01324	.01331	.01341	.01356	.01377	.01404	.01439	.01484	.01542	.01620	.01726	01878
	.02109	.02511	.03423	.09706										
47	.08454	.04851	.03767	.03192	.02820	.02555	.02354	.02195	.02065	.01957	.01866	.01787	.01719	.01660
	.01608	.01562	.01521	.01484	.01452	.01424	.01398	.01376	.01356	.01340	.01325	.01314	.01304	01298
	.01293	.01292	.01293	.01297	.01305	.01316	.01332	.01353	.01380	.01415	.01461	.01520	.01597	.01703
	.01854	.02084	.02483	.03389	.09632									
48	.08352	.04794	.03724	.03156	.02789	.02527	.02328	.02170	.02042	.01935	.01845	.01767	.01700	.01641
	.01589	.01543	.01502	.01466	.01433	.01405	.01379	.01356	.01336	.01319	.01304	.01292	.01282	.01274
	.01268	.01265	.01265	.01267	.01272	.01280	.01292	.01309	.01330	.01358	.01393	.01439	.01498	.01575
	.01681	.01831	.02055	.02456	.03356	09561								
49	.08253	.04740	.03683	03121	.02758	.02499	.02302	.02147	.02020	.01914	.01824	.01747	.01680	.01622
	.01570	.01525	.01484	.01448	.01415	.01386	.01361	.01338	.01317	.01300	.01284	.01271	.01260	.01251
	.01245	01241	.01238	.01239	.01242	.01248	.01257	.01269	01286	.01308	.01336	.01372	.01416	.01477
	.01554	.01659	.01808	.02036	.02430	.03324	.09491							
50	.08157	.04687	.03642	03087	.02729	.02472	.02278	.02124	.01998	01893	01805	.01728	.01662	.01604
	.01553	.01507	.01466	.01430	.01398	.01369	.01343	.01320	.01299	.01281	.01265	.01252	.01240	.01231
	.01223	.01218	.01214	.01213	.01215	.01218	.01225	.01235	.01248	.01265	.01288	.01316	.01352	01398
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9 .19879 .12090 .09669 .08477 .07852 .07628 .07857 .09047 .17505

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11 .17549 .10700 .08531 .07414 .06754 .06367 .06191 .06238 .06623 .07846 .15788

12 .16627 .10150 .08089 .07013 .06361 .05953 .05720 .05644 .05755 .06176 .07395 .15116

13 .15821 .09672 .07706 .06670 .06032 .05616 .05354 .05216 .05205 .05360 .05804 .07012 .14532

14 .15109 .09250 .07369 .06372 .05749 .05334 .05057 .04886 .04812 .04846 .05030 .05487 .06682 .14018

15 .14475 .08874 .07070 .06109 .05503 .05092 .04808 .04619 .04509 .04479 .04544 .04749 .05214 .06394
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16 .13906 .08536 .06802 .05874 .05285 .04881 .04595 .04395 .04265 .04198 .04200 .04288 .04507 .04977
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18 .12924 .07952 .06341 .05472 .04916 .04528 .04245 .04038 .03887 .03784 .03725 .03713 .03756 .03873
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19 .12500 .07698 .06140 .05298 .04757 .04377 .04099 .03890 .03735 .03623 .03550 .03514 .03520 .03576
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23 .11090 .06861 .05481 .04730 .04242 .03895 .03635 .03434 .03276 .03152 .03054 .02981 .02928 .02897
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26 .10280 .06371 .05095 .04399 .03944 .03619 .03373 .03181 .03027 .02903 .02803 .02721 .02657 .02608
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28 .09811 .06092 .04876 .04210 .03775 .03463 .03226 .03040 .02890 .02768 .02667 .02585 .02517 .02462
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	.02016	.01972	.01935	.01904	.01879	.01859	.01844	.01834	.01831	.01833	.01842	.01859	.01886	01924
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	.01979	.01935	.01898	.01866	.01839	.01817	.01801	.01789	.01782	.01781	.01785	.01796	.01815	.01842
	.01882	.01936	.02010	.02115	.02266	.02499	.02902	.03798	.09053					
38	.08102	.05061	.04063	.03515	.03154	.02892	.02692	.02532	.02402	.02293	.02201	.02122	.02054	.01996
	.01945	.01901	.01862	.01830	.01802	.01779	.01761	.01747	.01738	.01733	.01734	.01740	.01753	.01773
	.01801	.01842	.01896	.01971	.02076	.02226	.02457	.02857	.03745	.08952				
39	.07971	.04982	.04001	.03461	.03106	.02848	.02651	.02494	.02365	.02257	.02166	.02088	.02021	.01963
	.01912	.01868	.01829	.01796	.01768	.01744	.01724	.01709	.01697	.01690	.01688	01690	.01698	.01712
	.01733	.01763	.01804	.01859	.01934	.02038	.02188	.02418	.02815	.03694	.08855			
40	.07845	.04905	.03941	.03410	.03060	.02806	.02612	.02457	.02330	.02224	.02134	.02056	.01989	.01931
	.01881	.01836	.01798	.01764	.01735	.01710	.01690	.01673	.01660	.01651	.01646	.01646	.01649	.01659
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	.01851	.01807	.01768	.01734	.01705	.01679	.01658	.01640	.01626	.01615	.01608	.01605	.01606	.01611
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	.01823	.01779	.01740	.01705	.01676	.01650	.01628	.01609	.01593	.01581	.01573	01568	.01566	.01568
	.01575	.01586	.01604	.01627	.01659	.01702	.01758	.01834	.01937	.02085	.02310	.02697	.03553	.08585
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	.01796	.01752	.01713	.01678	.01648	.01622	.01599	.01580	.01563	.01550	.01540	.01533	.01530	.01530
	.01533	.01541	.01553	.01571	.01596	.01628	.01671	.01728	.01803	.01907	.02053	.02277	.02661	.03510
	.08501													
44	.07389	.04628	.03722	.03222	.02893	.02654	.02470	.02323	.02202	.02101	.02015	.01941	.01876	.01820
	.01770	.01726	.01687	.01653	.01622	.01596	.01572	.01552	.01535	.01521	.01510	.01502	.01496	.01494
	.01495	.01500	.01509	.01522	.01541	.01566	.01599	.01642	.01699	.01774	.01877	.02023	.02246	.02627
	.03469	.08421												
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	.02594	.03429	.08343											
46	.07185	.04504	.03623	.03138	.02818	.02585	.02406	.02263	.02145	.02047	.01962	.01890	.01826	.01771
	01721	.01678	.01639	01605	.01574	.01547	.01523	.01502	.01484	.01468	.01455	.01445	.01437	.01431
	.01429	.01429	.01432	.01439	.01449	.01464	.01484	.01510	.01544	.01588	.01645	.01720	.01822	.01967
	.02186	.02562	.03390	.08267										
47	.07089	.04445	.03577	.03099	.02782	.02553	.02376	.02235	.02118	.02021	.01937	.01865	.01803	.01747
	.01699	.01655	.01617	.01582	.01552	.01524	.01500	.01479	.01460	.01444	.01430	.01419	.01410	.01403
	.01399	.01398	.01399	.01403	.01410	.01422	.01437	.01458	.01484	.01518	.01563	.01620	.01695	.01797
	.01940	.02158	.02531	.03353	.08194									
48	.06995	.04388	.03532	.03060	02748	.02521	.02347	.02207	.02092	.01996	.01913	01842	.01780	.01725
	.01677	.01634	.01595	.01561	.01530	.01503	.01478	.01457	.01438	.01421	.01407	.01395	.01385	.01377
	.01372	.01369	.01368	.01370	.01375	.01384	.01395	.01411	.01432	.01459	.01494	.01538	.01595	.01670
	.01772	.01915	.02131	.02501	.03317	.08124								
49	.06905	.04333	.03488	.03023	.02715	.02491	.02319	.02181	.02067	.01972	.01890	.01820	.01758	.01704
	.01656	.01613	.01575	.01540	.01510	.01482	.01458	01436	.01416	.01399	.01384	.01372	.01361	.01353
	.01346	.01342	.01340	.01341	.01343	.01349	.01358	.01370	.01387	.01408	.01436	.01471	.01515	.01572
	.01647	.01748	.01890	.02105	.02473	.03283	.08055							
50	.06818	.04280	.03446	.02987	.02683	.02462	.02291	.02155	.02043	.01948	.01868	.01798	.01737	.01683
	.01635	.01593	.01555	.01521	.01490	.01462	.01438	.01416	.01396	.01379	.01363	.01350	.01339	.01330
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36	.07135	.04835	.04016	.03550	.03236	.03005	.02825	.02681	.02562	.02462	.02378	.02305	.02242	.02188
	.02141	.02101	.02066	.02037	.02013	.01994	.01980	.01972	.01968	.01971	.01979	.01996	.02021	.02057
	.02108	.02178	.02276	.02417	.02634	.03005	.03806	.07860						
37	.07003	.04747	.03945	.03487	.03179	.02952	.02776	.02634	.02517	.02418	.02335	.02263	.02201	.02147
	.02100	.02059	.02024	.01994	.01969	.01949	.01933	.01922	.01916	.01915	.01919	.01929	.01947	.01973
	.02010	.02062	.02132	.02230	.02371	.02587	.02954	.03746	.07756					
38	.06877	.04664	.03876	.03427	.03125	.02902	.02728	.02589	.02474	.02377	.02294	.02223	.02161	.02107
	.02060	.02020	.01984	.01954	.01928	.01907	.01890	.01877	.01868	.01864	.01864	.01870	.01882	.01901
	.01929	.01967	.02018	.02089	.02187	.02327	.02541	.02905	.03689	.07656				
39	.06756	.04584	.03811	.03370	.03073	.02853	.02683	.02546	.02432	.02337	.02255	.02185	.02123	.02070
	.02023	.01983	.01947	.01916	.01890	.01867	.01849	.01834	.01824	.01817	.01815	.01817	.01825	.01838
	.01858	.01886	.01925	.01977	.02048	.02146	.02285	.02498	.02858	.03634	.07561			
40	.06641	.04507	.03748	.03315	.03023	.02807	.02640	.02504	.02393	.02298	.02218	.02148	.02087	.02035
	.01988	.01947	.01912	.01880	.01853	.01830	.01811	.01795	.01783	.01775	.01770	.01769	.01773	.01782
	.01796	.01817	.01846	.01886	.01938	.02009	.02107	.02246	.02456	.02814	.03582	.07469		
41	.06530	.04434	.03688	.03262	.02975	.02763	.02598	.02465	.02355	.02262	.02182	.02113	.02053	.02001
	.01954	.01914	.01878	.01846	.01819	.01795	.01775	.01759	.01745	.01735	.01729	.01726	01727	.01732
	.01742	.01757	.01779	.01809	.01849	.01901	.01972	.02070	.02208	.02417	.02771	.03531	.07381	
42	.06424	.04363	.03630	.03211	.02929	.02720	.02558	.02427	.02318	.02227	.02148	.02080	.02021	.01968
	.01922	.01882	.01846	.01814	.01787	.01763	.01742	.01724	.01710	.01699	.01690	.01686	.01684	.01686
	.01693	.01704	.01720	.01742	.01773	.01813	.01866	.01937	.02034	.02172	.02379	.02730	.03483	.07296
43	.06321	.04295	.03574	.03163	.02885	.02679	.02520	.02390	.02283	.02193	.02115	.02048	.01989	.01937
	.01892	.01851	.01816	.01784	.01756	.01731	.01710	.01692	.01677	.01664	.01655	.01648	.01645	.01645
	.01648	.01656	.01668	.01685	.01708	.01739	.01779	.01833	.01904	.02000	.02137	.02343	.02691	.03437
	.07214													
44	.06223	.04230	.03520	.03116	.02842	.02640	.02483	.02355	.02250	.02160	.02084	.02017	.01959	.01908
	.01863	.01822	.01787	.01755	.01727	.01702	.01680	.01661	.01646	.01632	.01622	.01614	.01609	.01607
	.01608	.01612	.01621	.01633	.01651	.01675	.01706	.01747	.01801	.01872	.01968	.02104	.02308	.02653
	.03392	.07135												
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	.01834	.01794	.01759	.01727	.01699	.01674	.01652	.01633	.01616	.01602	.01591	.01582	.01575	.01572
	.01571	.01573	.01578	.01588	.01601	.01619	.01644	.01676	.01717	.01770	.01841	.01937	.02072	.02275
	.02617	.03349	.07059											
46	.06037	.04106	.03419	.03027	.02761	.02565	.02413	.02289	.02186	.02099	.02024	.01959	.01902	.01852
	.01808	.01768	.01732	.01701	.01672	.01647	.01625	.01605	.01588	.01574	.01561	.01552	.01544	.01539
	.01537	.01537	.01540	.01546	.01556	.01570	.01589	.01614	.01646	.01688	.01741	.01812	.01907	.02042
	.02243	.02582	.03308	.06985										
47	.05949	.04048	.03371	.02984	.02723	.02530	.02379	.02257	.02156	.02070	.01996	.01932	.01876	.01826
	.01782	.01742	.01707	.01675	.01647	.01622	.01599	.01579	.01562	.01547	01534	.01523	.01515	.01509
	.01505	.01503	.01504	.01508	.01515	.01526	.01541	.01560	.01586	.01618	.01660	.01713	.01784	.01879
	.02012	.02212	.02549	.03268	.06914									
46	.05864	.03991	.03324	.02944	.02686	.02496	.02347	.02227	.02127	.02042	.01969	.01906	.01850	.01801
	.01757	.01718	.01682	.01651	.01623	.01597	.01575	.01555	01537	.01521	.01508	.01496	.01487	.01480
	.01475	.01472	.01472	.01474	.01479	.01486	.01498	.01513	.01533	.01558	.01591	.01633	.01687	.01757
	.01852	.01984	.02183	.02517	.03230	.06846								
49	.05782	.03937	.03279	.02904	.02651	.02463	.02316	.02198	.02099	.02015	.01943	.01880	.01825	.01776
	01733	.01694	.01659	.01628	.01599	.01574	.01551	.01531	.01513	.01497	01483	.01471	.01461	.01453
	.01447	.01443	.01441	.01442	.01445	.01450	.01459	.01470	.01486	.01506	.01532	.01565	.01607	.01661
	.01731	.01825	.01957	.02154	.02485	.03193	.06779							
50	.05703	.03884	.03236	.02866	.02616	.02431	.02286	.02169	.02072	.01989	.01918	01856	.01801	01753
	.01710	.01671	.01636	.01605	.01577	.01552	.01529	.01508	.01490	.01474	.01460	.01447	.01437	.01428
	.01421	.01416	.01413	.01412	.01413	.01417	.01423	.01432	.01444	.01461	.01481	.01508	01541	.01583
	.01636	.01706	.01800	.01931	.02127	.02455	.03157	.06715						

K^* = .75000
N = 50

n

1 1.00000

2 .51937 .48063

3 .37736 .28403 .33861

4 .30370 .22096 .20508 .27026

5 .25737 .18534 .16392 .16415 .22922

6 .22506 .16152 .14031 .13289 .13875 .20147

7 .20104 .14415 .12420 .11510 .11300 .12130 .18125

8 .18236 .13078 .11220 .10280 .09855 .09914 .10840 .16576

9 .16734 .12010 .10280 .09359 .08860 .08680 .08878 .09857 .15344

10 .15497 .11130 .09514 .08632 .08111 .07837 .07795 .08074 .09071 .14336

11 .14458 .10400 .08879 .08036 .07516 .07202 .07059 .07102 .07430 .08427 .13494

12 .13570 .09769 .08339 .07536 .07027 .06698 .06507 .06445 .06543 .06900 .07890 .12776

13 .12801 .09225 .07875 .07109 .06614 .06282 .06067 .05954 .05947 .06081 .06456 .07434 .12160

14 .12130 .08749 .07469 .06738 .06259 .05930 .05704 .05563 .05502 .05533 .05691 .06077 .07040 .11620

15 .11530 .08328 .07111 .06412 .05950 .05626 .05397 .05241 .05150 .05126 .05183 .05359 .05751 .06697 .11140

16 .11000 .07953 .06792 .06122 .05677 .05361 .05131 .04967 .04859 .04804 .04807 .04883 .05071 .05465 .06395 .10710

17 .10530 .07615 .06505 .05863 .05434 .05126 .04898 .04731 .04612 .04538 .04509 .04532 .04623 .04819 .05213 .06126 .10330

18 .10100 .07310 .06247 .05630 .05215 .04915 .04692 .04523 .04399 .04313 .04265 .04255 .04293 .04394 .04596 .04989 .05885 .09982

19 .09706 .07033 .06012 .05418 .05017 .04726 .04507 .04339 .04212 .04119 .04057 .04028 .04033 .04082 .04192 .04397 .04788 .05668 .09667

20 .09349 .06780 .05797 .05225 .04837 .04554 .04339 .04173 .04045 .03947 .03878 .03835 .03820 .03838 .03895 .04011 .04219 .04607 .05471 .09379

21 .09022 .06547 .05600 .05047 .04672 .04397 .04187 .04024 .03895 .03795 .03720 .03669 .03640 .03637 .03664 .03728 .03848 .04058 .04443 .05291 .09114

22 .08720 .06333 .05418 .04884 .04520 .04253 .04048 .03887 .03759 .03658 .03580 .03522 .03485 .03468 .03474 .03508 .03578 .03701 .03912 .04293 .05127 .08870

23 .08442 .06134 .05250 .04733 .04380 .04121 .03920 .03762 .03636 .03534 .03454 .03392 .03348 .03322 .03314 .03328 .03367 .03442 .03568 .03778 .04155 .04975 .08644

24 .08183 .05950 .05094 .04593 .04251 .03998 .03802 .03647 .03522 .03421 .03339 .03275 .03227 .03194 .03176 .03176 .03196 .03240 .03318 .03446 .03656 .04029 .04835 .08434

25 .07942 .05778 .04949 .04462 .04130 .03884 .03693 .03541 .03417 .03317 .03235 .03169 .03117 .03079 .03055 .03045 .03051 .03076 .03124 .03205 .03334 .03543 .03912 .04705 .08238

26 .07717 .05618 .04813 .04341 .04017 .03777 .03591 .03442 .03320 .03221 .03139 .03072 .03018 .02977 .02948 .02931 .02927 .02938 .02966 .03018 .03101 .03230 .03439 .03803 .04584 .08054

27 .07507 .05468 .04686 .04226 .03912 .03678 .03496 .03350 .03230 .03132 .03050 .02982 .02927 .02884 .02851 .02828 .02817 .02819 .02834 .02866 .02920 .03005 .03135 .03342 .03702 .04471 .07882

28 .07310 .05327 .04566 .04119 .03813 .03585 .03407 .03264 .03146 .03049 .02968 .02900 .02844 .02798 .02763 .02737 .02721 .02714 .02720 .02739 .02774 .02830 .02916 .03046 .03252 .03608 .04366 .07721

29 .07124 .05194 .04454 .04018 .03719 .03497 .03323 .03183 .03068 .02972 .02891 .02823 .02767 .02720 .02682 .02654 .02633 .02622 .02620 .02629 .02651 .02688 .02746 .02833 .02964 .03169 .03520 .04266 .07568

30 .06950 .05069 .04348 .03923 .03632 .03414 .03244 .03107 .02994 .02899 .02819 .02752 .02695 .02648 .02609 .02578 .02554 .02539 .02532 .02534 .02546 .02570 .02609 .02669 .02756 .02887 .03090 .03438 .04173 .07424

31 .06785 .04951 .04248 .03834 .03549 .03336 .03170 .03035 .02924 .02831 .02752 .02685 .02628 .02580 .02540 .02508 .02482 .02464 .02452 .02449 .02454 .02468 .02494 .02536 .02596 .02684 .02815 .03016 .03360 .04085 .07288

32 .06628 .04839 .04153 .03749 .03470 .03263 .03100 .02968 .02859 .02767 .02689 .02623 .02566 .02517 .02477 .02443 .02416 .02395 .02380 .02372 .02372 .02379 .02396 .02424 .02467 .02529 .02617 .02747 .02947 .03287 .04003 .07159

33 .06480 .04733 .04063 .03668 .03396 .03193 .03033 .02904 .02797 .02706 .02629 .02564 .02507 .02459 .02417 .02383 .02354 .02331 .02314 .02303 .02299 .02301 .02311 .02329 .02359 .02403 .02465 .02554 .02683 .02882 .03218 .03924 .07036

34 .06340 .04633 .03977 .03591 .03325 .03126 .02970 .02843 .02738 .02649 .02573 .02508 .02452 .02403 .02362 .02327 .02297 .02273 .02254 .02240 .02232 .02230 .02235 .02246 .02267 .02298 .02343 .02406 .02495 .02623 .02820 .03153 .03850 .06920

35 .06207 .04537 .03896 .03518 .03258 .03063 .02910 .02786 .02682 .02595 .02520 .02456 .02400 .02351 .02310 .02274 .02244 .02218 .02198 .02182 .02172 .02166 .02167 .02173 .02186 .02208 .02241 .02286 .02350 .02439 .02567 .02762 .03091 .03779 .06809

36	.06080	.04446	.03819	.03449	.03194	.03003	.02853	.02731	.02629	.02543	.02469	.02406	.02350	.02302
	.02261	.02225	.02194	.02167	.02146	.02129	.02116	.02108	.02105	.02107	.02115	.02130	.02153	.02187
	.02233	.02297	.02386	.02513	.02707	.03032	.03712	.06703						
37	.05959	.04359	.03745	.03383	.03133	.02946	.02799	.02679	.02579	.02494	.02421	.02358	.02304	.02256
	.02214	.02178	.02147	.02120	.02097	.02079	.02065	.02055	.02049	.02048	.02052	.02061	.02078	.02102
	.02136	.02183	.02247	.02336	.02462	.02654	.02976	.03648	.06601					
38	.05843	.04276	.03674	.03320	.03075	.02891	.02747	.02629	.02531	.02447	.02376	.02314	.02259	.02212
	.02170	.02134	.02102	.02075	.02052	.02032	.02017	.02005	.01997	.01993	.01994	.02000	.02011	.02028
	.02053	.02088	.02135	.02199	.02288	.02414	.02604	.02923	.03587	.06504				
39	.05733	.04196	.03607	.03259	.03019	.02839	.02697	.02581	.02485	.02403	.02332	.02271	.02217	.02170
	.02129	.02092	.02061	.02033	.02009	.01989	.01972	.01959	.01950	.01944	.01942	.01944	.01951	.01963
	.01981	.02007	.02042	.02090	.02155	.02243	.02368	.02557	.02872	.03528	.06411			
40	.05627	.04120	.03542	.03201	.02966	.02789	.02650	.02536	.02441	.02360	.02290	.02230	.02177	.02130
	.02089	.02053	.02021	.01993	.01969	.01948	.01930	.01916	.01905	.01898	.01893	.01893	.01896	.01904
	.01918	.01937	.01964	.02000	.02048	.02112	.02200	.02325	.02512	.02824	.03473	.06322		
41	.05526	.04048	.03480	.03146	.02914	.02741	.02604	.02492	.02399	.02319	.02251	.02191	.02138	.02092
	.02051	.02015	.01983	.01955	.01931	.01909	.01891	.01876	.01864	.01855	.01849	.01846	.01847	.01852
	.01861	.01875	.01895	.01922	.01959	.02007	.02072	.02159	.02283	.02469	.02778	.03419	.06237	
42	.05429	.03978	.03421	.03092	.02865	.02695	.02560	.02450	02358	.02280	.02212	.02153	.02102	.02056
	.02015	.01979	.01947	.01919	.01894	.01873	.01854	.01838	.01825	.01815	.01807	.01803	.01802	.01804
	.01810	.01820	.01835	.01855	.01883	.01920	.01968	.02033	.02120	.02243	.02427	.02733	.03368	.06155
43	.05336	.03911	.03364	.03041	.02818	.02651	.02518	.02410	.02320	.02243	.02176	.02118	.02066	.02021
	.01981	.01945	.01913	.01885	.01860	.01838	.01819	.01802	.01789	.01777	.01769	.01763	.01760	.01760
	.01763	.01770	.01781	.01796	.01818	.01846	.01883	.01932	.01996	.02083	.02206	.02388	.02691	.03319
	.06076													
44	.05246	.03846	.03309	.02992	.02773	.02608	.02478	.02372	.02283	.02207	.02141	.02083	.02033	.01988
	.01948	.01912	.01881	.01853	.01827	.01805	.01786	.01769	.01754	.01742	.01733	.01726	.01721	01719
	.01720	.01724	.01732	.01744	.01760	.01782	.01811	.01848	.01897	.01961	.02048	.02169	.02350	.02650
	.03272	.06000												
45	.05160	.03784	.03256	.02945	.02729	.02567	.02439	.02335	.02247	.02172	.02107	.02050	.02000	.01956
	.01916	.01881	.01850	.01821	.01796	.01774	.01754	.01737	.01722	.01709	.01699	.01691	.01685	.01681
	.01681	.01683	.01688	.01696	.01708	.01725	.01748	.01777	.01814	.01863	.01927	.02014	.02134	.02314
	.02611	.03226	.05926											
46	.05077	.03724	.03205	.02899	.02687	.02528	.02402	.02299	.02212	.02139	.02075	.02019	.01969	.01925
	.01886	.01851	.01820	.01792	.01766	.01744	.01724	.01706	.01691	.01678	.01667	.01658	.01651	.01646
	.01644	.01644	.01647	.01653	.01662	.01675	.01692	.01715	.01745	.01782	.01831	.01895	.01981	.02101
	.02279	.02573	.03183	.05856										
47	04998	.03667	.03156	.02855	.02647	.02490	.02366	.02265	.02179	.02107	.02043	.01988	.01939	.01896
	.01857	.01822	.01791	.01763	.01738	.01715	.01695	.01677	.01662	.01648	.01636	.01627	.01619	.01613
	.01610	.01609	.01610	.01613	.01620	.01630	.01643	.01661	.01684	.01714	.01752	.01801	.01864	.01950
	.02069	.02245	.02537	03141	.05787									
48	04921	.03611	.03109	.02813	.02608	.02453	.02331	.02231	.02147	.02076	.02013	.01959	.01911	.01868
	.01829	.01795	.01764	.01736	.01711	.01688	.01668	.01650	.01634	.01620	.01608	.01597	.01589	.01582
	.01578	.01575	.01575	.01577	.01581	01588	.01599	.01613	.01631	.01654	.01684	.01722	.01771	.01835
	.01920	.02038	.02213	.02503	.03100	.05721								
49	.04847	.03556	.03064	02772	.02570	.02418	.02298	.02199	.02117	.02046	.01984	.01931	.01883	.01840
	.01802	.01768	.01737	.01710	.01685	.01662	.01642	.01623	.01607	.01593	.01580	.01569	.01560	.01553
	.01548	.01544	.01542	.01543	.01545	.01550	.01558	.01569	01584	.01602	.01626	.01656	01694	.01743
	.01807	01891	.02009	.02182	.02469	.03061	.05658							
50	.04775	.03506	.03020	.02732	.02533	.02384	.02265	.02168	.02087	.02017	.01956	01903	01856	01814
	.01777	.01743	.01712	.01684	.01660	.01637	.01617	.01598	.01582	.01567	.01554	.01543	.01533	.01526
	01519	.01515	.01512	.01511	.01512	.01516	.01521	.01530	.01541	.01556	.01575	.01599	.01629	.01667
	.01716	.01779	.01863	.01980	.02152	.02437	.03023	.05596						

K^{*} = .80000
N = 50

n	
1	1.00000
2	.51505 .48495
3	.36741 .29529 .33731
4	.29144 .22789 .21504 .26563
5	.24412 .18929 .17184 .17216 .22259
6	.21142 .16348 .14628 .14012 .14514 .19356
7	.18730 .14472 .12863 .12100 .11940 .12641 .17249
8	.16868 .13036 .11550 .10770 .10410 .10470 .11260 .15641
9	.15381 .11890 .10510 .09760 .09340 .09189 .09362 .10190 .14367
10	.14163 .10960 .09680 .08958 .08522 .08290 .08257 .08499 .09341 .13329
11	.13145 .10180 .08987 .08301 .07868 .07603 .07482 .07520 .07803 .08645 .12460
12	.12280 .09519 .08401 .07750 .07328 .07051 .06888 .06837 .06922 .07230 .08064 .11730
13	.11530 .08949 .07897 .07280 .06871 .06592 .06410 .06314 .06309 .06425 .06748 .07570 .11100
14	.10880 .08451 .07459 .06872 .06479 .06203 .06013 .05892 .05841 .05868 .06006 .06337 .07144 .10550
15	.10310 .08013 .07073 .06515 .06137 .05867 .05674 .05541 .05463 .05443 .05493 .05646 .05982 .06773 .10070
16	.09803 .07624 .06731 .06198 .05835 .05573 .05380 .05241 .05148 .05101 .05104 .05171 .05334 .05672 .06447 .09637
17	.09349 .07276 .06425 .05916 .05567 .05312 .05122 .04980 .04879 .04816 .04791 .04811 .04890 .05060 .05398 .06157 .09253
18	.08939 .06962 .06150 .05662 .05326 .05079 .04892 .04750 .04645 .04571 .04530 .04522 .04555 .04643 .04818 .05154 .05898 .08906
19	.08568 .06677 .05900 .05432 .05109 .04870 .04687 .04546 .04438 .04358 .04306 .04280 .04286 .04329 .04424 .04602 .04935 .05664 .08591
20	.08230 .06418 .05672 .05223 .04911 .04680 .04501 .04362 .04253 .04170 .04111 .04074 .04061 .04077 .04127 .04228 .04407 .04738 .05452 .08304
21	.07921 .06180 .05464 .05031 .04730 .04506 .04333 .04196 .04087 .04002 .03938 .03894 .03870 .03867 .03890 .03947 .04051 .04232 .04559 .05259 .08041
22	.07637 .05962 .05273 .04855 .04565 .04348 .04179 .04044 .03936 .03851 .03784 .03735 .03702 .03688 .03693 .03723 .03784 .03891 .04073 .04396 .05083 .07799
23	.07376 .05761 .05096 .04693 .04412 .04201 .04037 .03905 .03799 .03713 .03645 .03592 .03554 .03531 .03525 .03537 .03572 .03637 .03746 .03927 .04247 .04920 .07575
24	.07133 .05574 .04932 .04543 .04271 .04066 .03906 .03777 .03673 .03587 .03518 .03463 .03421 .03393 .03378 .03378 .03395 .03434 .03502 .03613 .03793 .04109 .04770 .07366
25	.06908 .05401 .04780 .04403 .04140 .03941 .03785 .03659 .03556 .03472 .03402 .03346 .03301 .03269 .03248 .03239 .03245 .03266 .03309 .03379 .03491 .03671 .03982 .04632 .07175
26	.06699 .05239 .04638 .04273 .04018 .03825 .03673 .03550 .03449 .03365 .03295 .03238 .03192 .03157 .03132 .03117 .03114 .03123 .03148 .03193 .03266 .03378 .03557 .03864 .04502 .06995
27	.06503 .05089 .04506 .04152 .03904 .03716 .03568 .03448 .03348 .03266 .03197 .03139 .03092 .03055 .03026 .03007 .02998 .02999 .03012 .03040 .03087 .03161 .03274 .03451 .03755 .04382 .06826
28	.06320 .04947 .04382 .04038 .03797 .03614 .03470 .03352 .03255 .03174 .03105 .03048 .03000 .02961 .02930 .02908 .02894 .02889 .02893 .02910 .02940 .02989 .03064 .03177 .03353 .03653 .04270 .06668
29	.06148 .04814 .04265 .03931 .03697 .03519 .03378 .03263 .03168 .03087 .03020 .02962 .02914 .02874 .02842 .02817 .02800 .02790 .02788 .02796 .02815 .02848 .02899 .02974 .03087 .03262 .03558 .04164 .06519
30	.05986 .04690 .04156 .03831 .03602 .03429 .03292 .03179 .03086 .03007 .02940 .02883 .02835 .02794 .02761 .02734 .02714 .02701 .02694 .02696 .02707 .02728 .02762 .02814 .02890 .03003 .03176 .03468 .04065 .06379
31	.05834 .04572 .04052 .03736 .03513 .03344 .03210 .03100 .03009 .02931 .02865 .02809 .02760 .02719 .02685 .02657 .02635 .02619 .02609 .02606 .02611 .02623 .02647 .02682 .02735 .02812 .02924 .03096 .03384 .03972 .06247
32	.05690 .04461 .03954 .03646 .03429 .03264 .03133 .03025 .02936 .02860 .02795 .02739 .02691 .02650 .02615 .02586 .02562 .02544 .02532 .02525 .02525 .02531 .02546 .02571 .02608 .02661 .02738 .02850 .03021 .03305 .03884 .06122
33	.05554 .04355 .03862 .03561 .03350 .03189 .03061 .02956 .02868 .02793 .02729 .02673 .02626 .02584 .02549 .02519 .02495 .02475 .02461 .02451 .02447 .02449 .02457 .02474 .02500 .02538 .02592 .02669 .02781 .02949 .03230 .03801 .06003
34	.05425 .04256 .03774 .03481 .03274 .03117 .02992 .02889 .02803 .02729 .02666 .02611 .02564 .02523 .02488 .02457 .02432 .02411 .02395 .02383 .02376 .02374 .02378 .02389 .02406 .02434 .02473 .02527 .02604 .02715 .02882 .03160 .03722 .05890
35	.05302 .04161 .03691 .03405 .03203 .03049 .02926 .02826 .02741 .02669 .02607 .02553 .02506 .02465 .02430 .02399 .02373 .02351 .02334 .02320 .02311 .02306 .02307 .02312 .02324 .02343 .02371 .02411 .02466 .02543 .02653 .02819 .03093 .03648 .05783

36	.05186	.04071	.03612	.03332	.03135	.02984	.02864	.02766	.02683	.02612	.02551	.02497	.02451	.02410
	.02375	.02344	.02318	.02295	.02276	.02262	.02251	.02244	.02241	.02243	.02250	.02263	.02284	.02313
	.02353	.02408	.02485	.02595	.02759	.03030	.03577	.05681						
37	.05075	.03985	.03536	.03263	.03070	.02923	.02805	.02709	.02627	.02557	.02497	.02445	.02399	.02359
	.02323	.02292	.02266	.02242	.02223	.02207	.02195	.02186	.02181	.02180	.02184	.02192	.02207	.02228
	.02257	.02298	.02354	.02431	.02539	.02702	.02970	.03509	.05583					
38	.04970	.03903	.03464	03197	.03008	.02864	.02749	.02654	.02574	.02506	.02447	.02395	.02349	.02309
	.02274	.02243	.02216	.02193	.02173	.02156	.02143	.02133	.02126	.02123	.02123	.02128	.02138	.02153
	.02175	.02205	.02246	.02302	.02379	.02487	.02648	.02912	.03445	.05490				
39	.04869	.03825	.03396	.03134	.02949	.02808	.02695	.02602	.02524	.02456	.02398	.02347	02302	.02263
	.02228	.02197	.02170	.02146	.02126	.02108	.02094	.02083	.02074	.02069	.02067	.02069	.02075	.02086
	.02102	.02125	.02156	.02197	.02253	.02329	.02437	.02596	.02858	.03384	.05401			
40	.04773	.03751	.03330	.03073	.02892	.02754	.02644	.02552	.02475	.02409	.02352	.02302	.02258	.02218
	.02184	.02163	.02126	.02102	.02081	.02063	.02048	.02036	.02026	.02020	.02016	.02016	.02019	.02026
	.02037	.02054	.02078	.02109	.02151	.02207	.02283	.02389	.02547	.02806	.03325	.05316		
41	.04681	.03679	.03267	.03016	.02838	.02703	.02595	.02505	.02429	.02364	.02308	.02258	.02215	.02176
	.02142	.02111	.02084	.02060	.02039	.02020	.02005	.01992	.01981	.01973	.01968	.01966	.01967	.01971
	.01979	.01991	.02009	.02033	.02065	.02107	.02162	.02238	.02344	.02500	.02756	.03269	.05234	
42	.04593	.03611	.03207	.02961	.02787	.02654	.02547	.02459	.02385	.02321	.02266	.02217	.02174	.02136
	.02101	.02071	.02044	.02020	.01999	.01980	.01964	.01950	.01939	.01930	.01924	.01920	.01919	.01921
	.01926	.01935	.01948	.01966	.01990	.02022	.02064	.02120	.02195	.02300	.02456	.02709	.03215	.05156
43	04508	.03545	.03149	.02908	.02737	.02607	.02502	.02416	.02343	.02280	.02225	.02177	.02135	.02097
	.02063	.02033	.02006	.01982	.01961	.01942	.01925	.01911	.01899	.01889	.01882	.01877	.01874	01874
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	.01790	.01792	.01796	.01804	.01814	.01829	.01849	.01874	.01907	.01949	.02005	.02079	.02182	.02332
	.02578	.03067	.04938											
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	.02295	.02537	.03021	.04871										
47	.04203	.03308	.02940	.02716	.02557	.02436	.02338	.02258	.02190	.02131	.02079	02034	.01994	.01958
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	.01894	.01865	.01839	.01816	.01794	.01775	.01758	.01742	.01728	.01716	01706	.01697	.01690	.01684
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	.01836	.01807	.01782	.01758	.01737	.01718	.01701	.01685	.01671	.01658	.01647	.01637	.01629	.01622
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15	.09230	.07683	.07006	.06589	.06300	.06091	.05939	.05834	.05772	.05756	.05797	.05921	.06189	.06804
	.09089													
16	.08744	.07283	.06642	.06246	.05970	.05767	.05616	.05506	.05432	.05395	.05397	.05452	.05583	.05853
	.06454	.08659												
17	.08311	.06927	.06318	.05941	.05676	.05480	.05332	.05220	.05139	.05089	.05069	.05086	.05150	.05287
	.05556	.06143	.08275											
18	.07923	.06606	.06027	.05667	.05414	.05224	.05079	.04967	.04883	.04825	.04791	.04785	.04812	.04884
	.05025	.05292	.05866	.07930										
19	.07572	.06317	.05764	.05420	.05177	.04994	.04852	.04742	.04656	.04593	.04551	.04531	.04535	.04570
	.04647	.04790	.05055	.05616	.07618									
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	.04114	.04160	.04244	.04389	.04648	.05185	.07075							
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25	.06015	.05030	.04596	.04324	.04129	.03980	.03861	.03764	.03684	.03617	.03562	.03517	.03482	.03456
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	.03314	.03302	.03299	.03307	.03328	.03364	.03423	.03514	.03656	.03898	.04381	.06052		
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	.03075	.03296	.03730	.05208										
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	.02676	.02652	.02633	.02617	.02605	.02598	.02594	.02596	.02603	.02616	.02638	.02669	.02713	.02775
	.02864	.02998	.03216	.03643	.05094									
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	.02608	.02584	.02564	.02547	.02534	.02524	.02519	.02517	.02521	.02529	.02544	.02566	.02598	.02642
	.02704	.02793	.02925	.03140	.03561	.04986								
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	.02544	.02520	.02499	.02481	.02467	.02456	.02449	.02445	.02446	.02450	.02460	.02476	.02499	.02531
	.02576	.02638	.02726	.02856	.03069	.03483	.04884							

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	.02484	.02459	.02438	.02420	.02405	.02393	.02384	.02379	.02377	.02378	.02384	.02395	.02411	.02435
	.02468	.02513	.02575	.02662	.02791	.03001	.03408	.04787						
37	.04328	.03631	.03324	.03131	.02992	.02884	.02797	.02724	.02663	.02609	.02563	.02522	.02486	.02454
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	.02021	.02100	.02215	.02396	.02743	.03903								
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	.02587	.02570	.02555	.02542	.02531	.02523	.02517	.02513	.02511	.02512	.02517	.02524	.02536	.02553
	.02577	.02609	.02653	.02715	.02805	.02947	.03214	.04012						
37	.03695	.03298	.03112	.02992	.02903	.02834	.02776	.02728	.02686	.02650	.02619	.02591	.02566	.02544
	.02524	.02507	.02492	.02479	.02468	.02459	.02452	.02447	.02444	.02443	.02445	.02450	.02459	.02471
	.02489	.02513	.02545	.02589	.02650	.02739	.02879	.03142	.03924					
38	.03608	.03221	.03040	.02922	.02836	.02768	.02712	.02665	.02624	.02589	.02558	.02530	.02506	.02484
	.02465	.02447	.02432	.02419	.02408	.02398	.02390	.02384	.02380	.02379	.02379	.02382	.02388	.02397
	.02410	.02427	.02451	.02484	.02527	.02588	.02676	.02814	.03072	.03841				
39	.03525	.03147	.02971	.02856	.02772	.02705	.02651	.02605	.02565	.02530	.02500	.02473	.02449	.02427
	.02408	.02391	.02376	.02362	.02351	.02341	.02333	.02326	.02321	.02318	.02317	.02318	.02322	02328
	.02338	.02351	.02369	.02394	.02426	.02469	.02529	.02616	.02753	.03007	.03762			
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	.02354	.02337	.02322	.02309	.02297	.02287	.02278	.02271	.02265	.02262	.02259	.02259	.02261	.02265
	.02272	.02282	.02296	.02314	.02339	.02371	.02414	.02473	.02559	.02694	.02944	.03687		
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	.02303	.02286	.02271	.02257	.02246	.02235	.02226	.02219	.02213	.02208	.02205	.02204	.02204	.02207
	.02212	.02219	.02229	.02243	.02262	.02286	.02318	.02361	.02420	.02505	.02638	.02884	.03615	
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	.02254	.02237	.02222	.02209	.02197	.02186	.02177	.02169	.02163	.02158	.02154	.02152	.02151	.02152
	.02155	.02161	.02168	.02179	.02193	.02212	.02237	.02269	.02311	.02370	.02453	.02585	.02827	.03546
43	.03230	.02885	.02724	.02620	.02543	.02482	.02432	.02390	.02354	.02322	.02294	.02269	.02246	.02226
	.02207	.02191	.02176	.02162	.02150	.02140	.02130	.02122	.02115	.02110	.02106	.02103	.02101	.02101
	.02103	.02106	.02112	.02120	.02131	.02146	.02165	.02189	.02221	.02263	.02321	.02404	.02533	.02772
	.03480													
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	.02163	.02146	.02132	.02118	.02106	.02096	.02086	.02078	.02071	.02065	.02060	.02056	.02054	.02053
	.02054	.02056	.02060	.02066	.02074	.02086	.02100	.02119	.02144	.02176	.02218	.02275	.02357	.02485
	.02720	.03416												
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	.02120	.02104	.02089	.02076	.02064	.02053	.02044	.02035	.02028	.02021	.02016	.02012	.02009	.02008
	.02007	.02008	.02011	.02015	.02022	.02030	.02042	.02057	02076	.02101	.02132	.02174	.02231	02312
	.02438	.02670	.03356											
46	.03041	.02717	.02566	.02468	.02396	.02339	.02292	.02253	.02218	.02188	.02162	.02138	.02116	.02097
	.02079	.02063-	.02049	.02036	.02024	.02013	.02003	.01994	.01987	.01980	.01975	.01970	.01967	.01965
	.01964	.01964	.01965	.01968	.01973	.01980	.01989	.02000	.02015	.02035	.02059	.02091	.02132	.02188
	.02268	.02393	.02622	.03297										
47	.02983	.02666	.02518	.02422	.02351	.02295	.02249	.02210	.02177	.02147	.02121	.02098	.02076	.02057
	.02040	.02024	.02010	.01997	.01985,	.01974	.01964	.01956	.01948	.01941	.01935	.01931	.01927	.01924
	.01922	.01921	.01922	.01924	.01927	.01933	.01940	.01949	.01961	.01976	.01995	.02019	.02051	.02092
	.02148	.02227	.02350	.02576	.03241									
48	.02927	.02616	.02471	.02377	.02308	.02253	.02208	.02170	.02137	.02108	.02082	.02059	.02038	.02019
	.02002	.01987	.01973	.01960	.01948	.01937	.01927	.01919	.01911	.01904	.01898	.01893	.01888	.01885
	.01883	.01882	.01881	.01882	.01885	.01888	.01894	.01901	.01911	.01923	.01938	.01957	.01981	.02013
	.02054	.02109	.02187	.02308	.02531	.03188								
49	.02874	.02569	.02426	.02334	.02266	.02213	.02168	.02131	.02098	.02070	.02045	.02022	.02002	.01983
	.01966	.01951	.01937	.01924	.01912	.01901	.01892	.01883	01875	.01868	.01862	.01856	.01852	.01848
	.01845	.01844	.01843	.01843	.01844	.01847	.01851	.01857	.01864	.01874	.01886	.01901	.01921	.01945
	.01976	.02017	.02071	.02149	.02269	.02489	.03136							
50	02822	.02523	.02383	.02293	.02226	.02174	.02130	.02093	.02062	.02034	.02009	.01986	.01966	.01948
	.01931	.01916	.01902	.01890	.01878	.01867	.01858	.01849	.01841	.01834	.01827	.01822	.01817	.01813
	.01810	.01808	.01806	.01806	.01806	.01808	.01811	.01815	.01821	.01829	.01839	.01851	.01866	.01886
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1 1.00000

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13 .08497 .08017 .07779 .07625 .07516 .07438 .07385 .07357 .07356 .07393 .07490 .07717 .08429

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19 .05934 .05606 .05442 .05333 .05252 .05190 .05140 .05100 .05069 .05045 .05029 .05022 .05024 .05038
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22 .05166 .04882 .04740 .04646 .04575 .04520 .04475 .04438 .04408 .04383 .04363 .04348 .04339 .04334
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23 .04953 .04681 .04546 .04455 .04388 .04335 .04291 .04256 .04226 .04201 .04181 .04165 .04154 .04147
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25 .04577 .04327 .04202 .04119 .04057 .04008 .03967 .03934 .03905 .03881 .03861 .03844 .03831 .03821
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26 .04410 .04170 .04050 .03970 .03910 .03863 .03824 .03791 .03763 .03740 .03720 .03703 .03689 .03679
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36	.03241	.03067	.02981	.02923	.02879	.02845	.02816	.02791	.02770	.02751	.02735	.02720	.02707	.02695
	.02685	.02676	.02668	.02661	.02656	.02651	.02648	.02646	.02645	.02646	.02648	.02652	.02659	.02668
	.02680	.02698	.02721	.02753	.02800	.02872	.03002	.03346						
37	.03158	.02989	.02904	.02848	.02806	.02772	.02744	.02720	.02699	.02681	.02665	.02651	.02638	.02626
	.02616	.02607	.02599	.02592	.02586	.02582	.02578	.02575	.02574	.02573	.02575	.02577	.02582	.02589
	.02598	.02611	.02628	.02651	.02683	.02729	.02800	.02927	.03264					
38	.03079	.02914	.02832	.02778	.02737	.02704	.02676	.02653	.02633	.02615	.02599	.02585	.02572	.02561
	.02551	.02542	.02534	.02527	.02521	.02516	.02512	.02509	.02506	.02506	.02506	.02507	.02510	.02515
	.02522	.02532	.02545	.02562	.02585	.02617	.02662	.02732	.02857	.03187				
39	.03004	.02844	.02764	.02711	.02671	.02638	.02612	.02589	.02569	.02552	.02536	.02522	.02510	.02499
	.02489	.02480	.02472	.02465	.02459	.02454	.02449	.02446	.02443	.02442	.02441	.02442	.02444	.02447
	.02452	.02459	.02469	.02482	.02499	.02522	.02554	.02598	.02667	.02790	.03114			
40	.02933	.02776	.02699	.02647	.02608	.02576	02550	.02528	.02509	.02492	.02477	.02463	.02451	.02440
	.02430	.02421	.02413	.02406	.02400	.02395	.02390	.02386	.02383	.02381	.02380	.02380	.02381	.02383
	.02387	.02392	.02400	.02410	.02423	.02440	.02463	.02494	.02538	.02606	.02726	.03044		
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	.02374	.02365	.02357	.02350	.02344	.02338	.02334	.02330	.02326	.02324	.02322	.02322	.02322	.02323
	.02326	.02330	.02336	.02343	.02353	.02366	.02383	.02406	.02437	.02480	.02547	.02666	.02977	
42	.02801	.02651	.02577	.02528	.02491	.02461	.02436	.02415	.02396	.02380	.02366	.02353	.02341	.02330
	.02321	.02312	.02304	.02297	.02291	.02285	.02280	.02276	.02273	.02270	.02268	.02267	.02266	.02267
	.02269	.02271	.02276	.02281	.02289	.02299	.02312	.02329	.02352	.02382	.02425	.02491	.02608	.02913
43	.02739	.02593	.02521	.02472	.02436	.02407	.02383	.02362	.02344	.02328	.02314	.02301	.02290	.02279
	.02270	.02261	.02253	.02246	.02240	.02234	.02229	.02225	.02221	.02218	.02216	.02214	.02214	.02214
	.02215	.02217	.02220	.02224	.02230	.02238	.02248	.02261	.02278	.02300	.02330	.02373	.02438	.02552
	.02853													
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	.02221	.02212	.02205	.02198	.02191	.02186	.02181	.02176	.02173	.02169	.02167	.02165	.02164	.02163
	.02163	.02165	.02167	.02170	.02175	.02181	.02189	.02199	.02212	.02229	.02251	.02281	.02323	.02387
	.02499	.02795												
45	.02624	.02484	.02415	.02369	.02334	.02306	.02283	.02263	.02246	.02231	.02217	.02205	.02194	.02184
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	.02338	.02449	.02739											
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	.02130	.02122	.02114	.02107	.02101	.02095	.02090	.02085	.02081	.02078	.02075	.02073	.02071	.02070
	.02069	.02069	.02070	.02072	.02074	.02078	.02083	.02089	.02097	.02107	.02120	.02137	.02159	.02188
	.02229	.02291	.02400	.02686										
47	.02518	.02384	.02318	.02274	.02241	.02214	.02192	.02173	.02156	.02141	.02128	.02116	.02106	.02096
	.02087	.02079	.02071	.02065	.02058	.02053	.02048	.02043	.02039	.02035	.02032	.02030	.02028	.02026
	.02025	.02025	.02025	.02026	.02028	.02031	.02035	.02040	.02046	.02054	.02064	.02077	.02094	.02116
	.02145	.02185	.02246	.02354	.02634									
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	.02046	.02038	.02031	.02024	.02018	.02012	.02007	.02002	01998	.01995	.01991	.01989	.01986	.01985
	.01983	.01983	.01983	.01983	.01984	.01986	.01989	.01993	01998	.02005	.02013	.02023	.02036	.02053
	.02074	.02103	.02143	.02203	.02309	.02585								
49	.02420	.02292	.02228	.02186	.02154	.02129	.02108	.02089	.02073	.02059	.02047	.02035	.02025	.02015
	.02007	.01999	.01992	.01985	.01979	.01973	.01968	.01963	.01959	.01955	.01952	.01949	.01947	.01945
	01943	.01943	.01942	.01942	.01943	.01944	01947	.01950	.01954	.01959	.01965	.01974	.01984	.01997
	.02013	.02034	.02063	.02102	.02162	.02266	.02538							
50	.02374	.02248	.02186	.02145	.02114	.02089	.02068	.02050	02034	.02021	.02008	.01997	.01987	.01977
	01969	.01961	.01954	.01947	.01941	.01936	.01931	.01926	.01922	.01918	.01915	.01912	.01909	.01907
	.01905	.01904	.01903	.01903	.01903	.01904	.01906	.01908	.01912	.01916	.01921	.01928	.01936	.01946
	.01959	.01975	.01996	.02024	.02063	.02122	.02225	.02493						

$K^* = .99000$
$N = 50$

n	
1	1.00000
2	.50068 .49932
3	.33484 .33167 .33349
4	.25182 .24908 .24843 .25068
5	.20191 .19962 .19872 .19876 .20099
6	.16858 .16665 .16577 .16543 .16573 .16784
7	.14474 .14307 .14227 .14185 .14176 .14218 .14414
8	.12683 .12537 .12460 .12420 .12400 .12410 .12450 .12636
9	.11290 .11160 .11090 .11050 .11030 .11020 .11030 .11080 .11250
10	.10170 .10050 .09995 .09957 .09932 .09919 .09917 .09932 .09981 .10140
11	.09256 .09150 .09096 .09061 .09036 .09021 .09014 .09016 .09034 .09083 .09233
12	.08493 .08396 .08346 .08313 .08290 .08274 .08264 .08261 .08267 .08286 .08334 .08476
13	.07846 .07757 .07711 .07681 .07658 .07643 .07632 .07626 .07626 .07633 .07654 .07700 .07834
14	.07292 .07209 .07167 .07138 .07117 .07101 .07090 .07083 .07080 .07082 .07091 .07111 .07156 .07284
15	.06811 .06734 .06694 .06667 .06648 .06633 .06621 .06613 .06608 .06607 06611 .06621 .06641 .06685 .06806
16	.06389 .06318 .06281 .06255 .06237 .06222 .06211 .06203 .06197 .06194 .06194 .06199 .06209 .06230 .06272 .06388
17	.06017 .05950 .05915 .05892 .05874 .05860 .05849 .05841 .05835 .05831 .05829 .05831 .05836 .05847 .05867 .05908 .06019
18	.05687 .05623 .05590 .05568 .05551 .05538 .05527 .05519 .05513 .05508 .05506 .05505 .05507 .05513 .05524 .05545 .05584 .05690
19	.05391 .05331 .05300 .05278 .05262 .05250 .05240 .05231 .05225 .05220 .05217 .05215 .05216 .05218 .05225 .05236 .05256 .05295 .05396
20	.05124 .05067 .05038 .05017 .05002 .04990 .04980 .04972 .04966 .04961 .04957 .04955 .04954 .04955 .04959 .04965 .04977 .04996 .05034 .05131
21	.04882 .04829 .04800 .04781 .04767 .04755 .04746 .04738 .04732 .04727 .04723 .04720 .04718 .04718 .04720 .04724 .04730 .04742 .04761 .04797 .04891
22	.04663 .04611 .04585 .04566 .04553 .04542 .04532 .04525 .04519 .04514 .04509 .04506 .04504 .04503 .04504 .04506 .04510 .04517 .04528 .04547 .04582 .04673
23	.04462 .04413 .04388 .04370 .04357 .04346 .04338 .04330 .04324 .04319 .04315 .04312 .04309 .04308 .04307 .04308 .04311 .04315 .04322 .04333 .04352 .04386 .04474
24	.04278 .04231 .04207 .04190 .04178 .04167 .04159 .04152 .04146 .04141 .04137 .04133 .04131 .04129 .04128 .04128 .04129 .04132 .04136 .04143 .04155 .04173 .04206 .04291
25	.04109 .04064 .04041 .04025 .04012 .04003 .03994 .03988 .03982 .03977 .03973 .03969 .03967 .03965 .03963 .03963 .03963 .03965 .03967 .03972 .03979 03990 .04008 .04041 .04122
26	.03953 .03909 .03887 .03872 .03860 .03850 .03843 .03836 .03830 .03826 .03821 .03818 .03815 .03813 .03811 .03810 .03810 .03811 .03813 .03816 .03820 .03827 .03838 .03856 .03888 .03967
27	.03808 .03766 .03745 .03730 .03719 .03710 .03702 .03696 .03690 .03685 .03681 .03678 .03675 .03673 .03671 .03670 .03669 .03669 .03670 .03672 .03675 .03680 .03687 .03698 .03715 .03746 .03823
28	.03673 .03633 .03612 .03598 .03587 .03579 .03571 .03565 .03560 .03555 .03551 .03548 .03545 .03542 .03540 .03539 .03538 .03538 .03538 .03539 .03541 .03545 .03549 .03557 .03567 .03584 .03614 .03689
29	.03548 .03509 .03489 .03476 .03465 .03457 .03450 .03444 .03438 .03434 .03430 .03427 .03424 .03421 .03419 .03418 .03416 .03416 .03416 .03416 .03418 .03420 .03423 .03428 .03435 .03446 .03462 .03492 .03564
30	.03431 .03394 .03374 .03361 .03351 .03343 .03336 .03330 .03325 .03321 .03317 .03314 .03311 .03308 .03306 .03304 .03303 .03302 .03302 .03302 .03303 .03304 .03306 .03310 .03315 .03322 .03332 .03348 .03377 .03448
31	.03321 .03285 .03267 .03254 .03244 .03236 .03230 .03224 .03219 .03215 .03211 .03208 .03205 .03202 .03200 .03199 .03197 .03196 .03195 .03195 .03196 .03196 .03198 .03200 03204 .03209 .03216 .03226 .03242 .03270 .03339
32	03219 .03184 .03166 03153 .03144 .03136 .03130 .03124 .03120 .03116 .03112 .03109 .03106 03103 .03101 .03099 .03098 .03097 .03096 .03095 03095 .03096 .03097 .03099 .03101 .03105 .03109 .03116 .03127 .03142 .03170 .03237
33	.03122 .03088 .03071 .03059 .03050 .03042 .03036 .03031 .03026 .03022 .03019 .03015 .03013 03010 .03008 .03006 .03005 .03003 .03002 .03002 .03002 .03002 .03002 .03003 .03005 .03008 .03011 .03016 .03023 .03033 03048 .03075 .03140
34	.03031 .02999 .02982 .02970 .02961 .02954 .02948 .02943 .02938 .02934 .02931 .02928 .02925 02922 .02920 .02918 .02917 .02916 .02915 .02914 02913 .02913 .02913 .02914 .02915 .02917 .02920 .02923 .02928 .02935 .02945 .02960 .02986 .03050
35	.02945 .02914 .02897 .02886 .02878 .02871 .02865 .02860 .02855 .02851 .02848 .02845 .02842 02840 .02838 .02836 .02834 .02833 .02832 .02831 .02830 .02830 .02830 .02830 .02831 .02832 02834 .02837 .02841 .02845 02852 .02862 .02877 .02902 .02965

36	.02865	.02834	.02818	.02807	.02799	.02792	.02786	.02781	.02777	.02773	.02770	.02767	.02764	.02762
	.02760	.02758	.02756	.02755	.02754	.02753	.02752	.02751	.02751	.02751	.02752	.02753	.02754	.02756
	.02759	.02762	.02767	.02774	.02784	.02798	.02823	.02884						
37	.02788	.02758	.02742	.02732	.02724	.02717	.02712	.02707	.02703	.02699	.02696	.02693	.02690	.02688
	.02686	.02684	.02682	.02681	.02680	.02679	.02678	.02677	.02677	.02677	.02677	.02678	.02679	.02680
	.02682	.02685	.02688	.02693	.02700	.02709	.02724	.02748	.02808					
38	.02715	.02686	.02671	.02661	.02653	.02647	.02641	.02637	.02633	.02629	.02626	.02623	.02620	.02618
	.02616	.02614	.02612	.02611	.02610	.02609	.02608	.02607	.02607	.02606	.02606	.02607	.02607	.02609
	.02610	.02612	.02615	.02618	.02623	.02630	.02639	.02653	.02677	.02735				
39	.02646	.02618	.02603	.02593	.02586	.02580	.02574	.02570	.02566	.02562	.02559	.02556	.02554	.02552
	.02550	.02548	.02546	.02545	.02543	.02542	.02541	.02541	.02540	.02540	.02540	.02540	.02540	.02541
	.02542	.02543	.02546	.02548	.02552	.02557	.02563	.02572	.02586	.02610	.02667			
40	.02581	.02553	.02539	.02529	.02522	.02516	.02511	.02506	.02502	.02499	.02496	.02493	.02491	.02489
	.02486	.02485	.02483	.02482	.02480	.02479	.02478	.02477	.02477	.02476	.02476	.02476	.02476	.02477
	.02478	.02479	.02480	.02482	.02485	.02489	.02493	.02500	.02509	.02522	.02546	.02601		
41	.02519	.02492	.02478	.02468	.02461	.02455	.02450	.02446	.02442	.02439	.02436	.02433	.02431	.02429
	.02427	.02425	.02423	.02422	.02420	.02419	.02418	.02417	.02417	.02416	.02416	.02416	.02416	.02416
	.02417	.02417	.02419	.02420	.02422	.02425	.02429	.02433	.02440	.02449	.02462	.02485	.02539	
42	.02459	.02433	.02419	.02410	.02403	.02397	.02393	.02388	.02385	.02381	.02379	.02376	.02374	.02371
	.02369	.02368	.02366	.02364	.02363	.02362	.02361	.02360	.02359	.02359	.02358	.02358	.02358	.02358
	.02359	.02359	.02360	.02361	.02363	.02365	.02368	.02371	.02376	.02382	.02391	.02404	.02427	.02480
43	.02403	.02377	.02364	.02355	.02348	.02342	.02338	.02333	.02330	.02327	.02324	.02321	.02319	.02317
	.02315	.02313	.02312	.02310	.02309	.02308	.02307	.02306	.02305	.02304	.02304	.02303	.02303	.02303
	.02303	.02304	.02305	.02305	.02307	.02308	.02310	.02313	.02317	.02321	.02328	.02336	.02349	.02371
	.02424													
44	.02349	.02324	.02311	.02302	.02295	.02290	.02285	.02281	.02278	.02274	.02272	.02269	.02267	.02265
	.02263	.02261	.02260	.02258	.02257	.02256	.02255	.02254	.02253	.02252	.02252	.02251	.02251	.02251
	.02251	.02251	.02252	.02252	.02253	.02255	.02256	.02258	.02261	.02265	.02269	.02275	.02284	.02297
	.02318	.02370												
45	.02297	.02273	.02260	.02251	.02245	.02239	.02235	.02231	.02228	.02225	.02222	.02219	.02217	.02215
	.02213	.02211	.02210	.02208	.02207	.02206	.02205	.02204	.02203	.02202	.02202	.02201	.02201	.02201
	.02201	.02201	.02201	.02202	.02203	.02204	.02205	.02207	.02209	.02211	.02215	.02219	.02226	.02234
	.02246	.02268	.02318											
46	.02248	.02224	.02211	.02203	.02197	.02191	.02187	.02183	.02180	.02177	.02174	.02172	.02169	.02167
	.02166	.02164	.02162	.02161	.02160	.02158	.02157	.02156	.02156	.02155	.02154	.02154	.02153	.02153
	.02153	.02153	.02153	.02154	.02154	.02155	.02156	.02157	.02159	.02161	.02164	.02167	.02172	.02178
	.02186	.02198	.02219	.02269										
47	.02200	.02177	.02165	.02157	.02150	.02145	.02141	.02137	.02134	.02131	.02128	.02126	.02124	.02122
	.02120	.02118	.02117	.02115	.02114	.02113	.02112	.02111	.02110	.02109	.02109	.02108	.02108	.02107
	.02107	.02107	.02107	.02107	.02108	.02108	.02109	.02110	.02112	.02113	.02116	.02118	.02122	.02126
	.02132	.02140	.02152	.02173	.02222									
48	.02155	.02132	.02120	.02112	.02106	.02101	.02097	.02093	.02090	.02087	.02085	.02082	.02080	.02078
	.02076	.02075	.02073	.02072	.02071	.02069	.02068	.02067	.02067	.02066	.02065	.02064	.02064	.02064
	.02063	.02063	.02063	.02063	.02064	.02064	.02065	.02065	.02067	.02068	.02070	.02072	.02075	.02078
	.02082	.02088	.02096	.02108	.02128	.02176								
49	.02111	.02089	.02077	.02070	.02064	.02059	.02055	.02051	.02048	.02045	.02043	.02040	.02038	.02036
	.02035	.02033	.02031	.02030	.02029	.02028	.02027	.02026	.02025	.02024	.02023	.02023	.02022	.02022
	.02021	.02021	.02021	.02021	.02021	.02022	.02022	.02023	.02024	.02025	.02026	.02028	.02030	.02033
	.02036	.02040	.02046	.02054	.02066	.02086	.02133							
50	.02070	.02048	.02036	.02029	.02023	.02018	.02014	.02010	.02007	.02005	.02002	.02000	.01998	.01996
	.01994	.01993	.01991	.01990	.01989	.01987	.01986	.01985	.01985	.01984	.01983	.01982	.01982	.01981
	.01981	.01981	.01981	.01981	.01981	.01981	.01981	.01982	.01982	.01983	.01984	.01986	.01988	.01990
	.01992	.01996	.02000	.02006	.02014	.02025	.02045	.02091						

Appendix V: Evaluation of moments of $EV_I(\xi,b)$

By definition, for $p > 0$

$$\psi(p) = \frac{\Gamma'(p)}{\Gamma(p)} .$$

But, for any positive integer k

$$\Gamma^{(k)}(p) = \int_0^\infty x^{p-1} \exp(-x)(\ln x)^k dx$$

and

$$\Gamma^{(k)}(1) = \lambda_1 = \int_0^\infty (\ln x)e^{-x} dx .$$

In general, define $\lambda_i = \int_0^\infty (\ln x)^i e^{-x} dx$. In particular $\lambda_1 = \Gamma^{(1)}(1)$ $= -\gamma = -.5772$. From Jahnke (1960) (p. 12)

$$\psi^{(1)}(z) = \sum_{k=0}^\infty \frac{1}{(z+k)^2}$$

and, by differentiating successively and setting $z = 1$,

$\psi^{(k)}(z) = (-1)^{k-1} k!\zeta(k+1)$, where $\zeta(x)$ is Riemann's zeta function.

Using values of $\zeta(k)$ [Jahnke (1960, p. 37)], we have

$$\psi^{(1)}(1) = \frac{\pi^2}{6}$$

$$\psi^{(2)}(1) = -2.404$$

$$\psi^{(3)}(1) = \frac{\pi^4}{15} \qquad .$$

Under the homogeneous model, we obtain

$$\mu_2 = E[\ell n\ X - E(\ell n\ X)]^2 = b^2(\lambda_2 - \lambda_1^2)$$

$$\mu_3 = b^3(\lambda_3 - 3\lambda_2\lambda_1 + 2\lambda_1^3) = -2.4036b^3$$

$$\mu_4 = b^4(\lambda_4 - 4\lambda_3\lambda_1 + 6\lambda_2\lambda_1^2 - 3\lambda_1^4) = 14.6119b^4$$

(see [Menon (1963)]. Here

$$\lambda_2 = \gamma^2 + \frac{\pi^2}{6}, \quad \lambda_3 = -5.4445, \quad \lambda_4 = 235601.$$

However, under the exchangeable model, using $Y = \ell n\ X$,

$$E_{het}(Y) = \frac{b\gamma}{n}\{n - 1 + \frac{1}{k^*}\}$$

$$E_{het}(Y^2) = \frac{\pi^2 b^2}{6n}\{n - 1 + \frac{1}{k^{*2}}\} + \{\frac{b\gamma}{n}\ (n - 1 + \frac{1}{k^*})\}^2$$

$$E_{het}(Y^3) = \frac{n-1}{n}(-b)^3\psi^{(2)}(1) + \frac{1}{n}\left(\frac{-b}{k*}\right)^3\psi^{(2)}(1)$$

$$= \frac{\psi^{(2)}(1)(-b^3)}{n}\left\{n - 1 + \frac{1}{k*^3}\right\}$$

$$E_{het}(Y^4) = \frac{n-1}{n}(b^4)\psi^{(3)}(1) + \frac{1}{n}\left(\frac{-b}{k*}\right)^4\psi^{(3)}(1)$$

$$= \frac{\psi^{(3)}(1)(b^4)}{n}\left\{n - 1 + \frac{1}{k*^4}\right\}$$

where $E_{\text{homogeneous}}(Y^r) = -b^r\psi^{(r)}(1)$ [see Patel et al (1976)].

Appendix VI Computer Programs

Program A was used to compute u(r,n,k*) for the Weibull distribution using the recursive formula from Chapter V.3, p. 103. Extended precision Fortran was used to perform the calculations.

Program A

```
      IMPLICIT LOGICAL*1 (A-Z)
      DIMENSION U(50,50), EFUN(501), UBAR(50)
      REAL*16 U, EFUN, UBAR, SUM, K, C, Y, YINCR
      INTEGER I, J, N, R
C
      WRITE(6,200)
 200  FORMAT(' Enter K (with decimal point) and N
     1      ' (integer)'/ each followed by a comma.')
      READ(5,100) K, N
 100  FORMAT(F7.5,I2)
      WRITE(7,201) K, N
 201  FORMAT('1K = ',F7.5,/' N = ',I2,/'0')
C
C INITIALIZATIONS
C
      DO 300 I=1,N
        DO 301 J=1,N
          U(J,I) = 0.0Q0
 301    CONTINUE
 300  CONTINUE
C
      YINCR = (1.Q0/500.Q0)
  DO 302 I=1,501
    Y = QFLOAT(I-1)*YINCR
    IF (Y .GT. 1.Q-30) GO TO 3021
    EFUN(I) = 0.0Q0
    GO TO 302
1   EFUN(I) = QEXP(-(QABS(QLOG(1.Q0/Y)))**(1.Q0/K))
  CONTINUE
CALCULATE VALUES OF U-BAR (FIRST COLUMN OF U).
USE SIMPSON'S (1/3) RULE FOR THE INTEGRATION.
  DO 310 R=1,N
    IF (EFUN(1) .LT. 1.Q-70) EFUN(1) = 1.Q-70
    SUM = EFUN(1)**(R-1)
    DO 311 J=1,249
      IF (EFUN(2*J) .LT. 1.Q-70) EFUN(2*J) = 1.Q-70
      IF (EFUN(2*J+1) .LT. 1.Q-70) EFUN(2*J+1) = 1.Q-70
      SUM = SUM + 4.Q0*EFUN(2*J)**(R-1)
 1               + 2.Q0*EFUN(2*J+1)**(R-1)
 311    CONTINUE
        SUM = SUM + 4.Q0*EFUN(500)**(R-1) + EFUN(501)**(R-1)
        UBAR(R) = SUM*(.002Q0/3.Q0)
 310  CONTINUE
C
C SET 1ST COLUMN OF U EQUAL TO UBAR
C
      DO 315 I=1,N
        U(I,1) = UBAR(I)
 315  CONTINUE
```

```
      WRITE(7,250) U(1,1)
 250  FORMAT(14('0',F8.5)/,3(3X,14(1X,F8.5)/)/)
C
C  CALCULATE THE REST OF U AND WRITE IT OUT.
C
      DO 320 I=2,N
        DO 321 R=2,I
          SUM = 0.Q0
          DO 322 J=1,R
            CALL COMB(R-1,J-1,C)
            SUM = SUM + C*((-1)**(J-1))*UBAR(I-R+J)
 322      CONTINUE
          CALL COMB(I-1,R-1,C)
          U(I,R) = C*SUM
 321    CONTINUE
        WRITE(7,250) (U(I,R),R=1,I)
 320  CONTINUE
      STOP
      END
      SUBROUTINE COMB(I,J,C)
      IMPLICIT REAL*16 (A-H,K,O-Z)
      REAL*16 K(15)
C
      DO 300 IK=1,15
        K(IK) = 0.Q0
 300  CONTINUE
C
      LSTOP = MIN0(I-J,J)
      L=1
      PN=QFLOAT(L)
      PD=QFLOAT(L)
      IF (LSTOP .LE. 0) GO TO 3999
      DO 3000 L=1,LSTOP
        A=QFLOAT(L)
        PD=A*PD
        M=I-L+1
        A=QFLOAT(M)
        PN=A*PN
        DO 3001 IK=1,15
          K(IK) = K(IK)+1
 3001   CONTINUE
        IF (K(1) .NE. 2.Q0) GO TO 3101
          FACT=2.Q0
          PN=PN/FACT
          PD=PD/FACT
          K(1) = 0.Q0
 3101   CONTINUE
        IF (K(2) .NE. 3.Q0) GO TO 3102
          FACT=3.Q0
          PN=PN/FACT
          PD=PD/FACT
          K(2) = 0.Q0
```

```
3102   CONTINUE
       IF (K(3) .NE. 5.Q0) GO TO 3103
         FACT=5.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(3) = 0.Q0
3103   CONTINUE
       IF (K(4) .NE. 7.Q0) GO TO 3104
         FACT=7.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(4) = 0.Q0
3104   CONTINUE
       IF (K(5) .NE. 11.Q0) GO TO 3105
         FACT=11.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(5) = 0.Q0
3105   CONTINUE
       IF (K(6) .NE. 13.Q0) GO TO 3106
         FACT=13.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(6) = 0.Q0
3106   CONTINUE
       IF (K(7) .NE. 17.Q0) GO TO 3107
         FACT=17.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(7) = 0.Q0
3107   CONTINUE
       IF (K(8) .NE. 19.Q0) GO TO 3108
         FACT=19.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(8) = 0.Q0
3108   CONTINUE
       IF (K(9) .NE. 23.Q0) GO TO 3109
         FACT=23.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(9) = 0.Q0
3109   CONTINUE
       IF (K(10) .NE. 29.Q0) GO TO 3110
         FACT=29.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(10) = 0.Q0
```

```
3110   CONTINUE
       IF (K(11) .NE. 31.Q0) GO TO 3111
         FACT=31.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(11) = 0.Q0
3111   CONTINUE
       IF (K(12) .NE. 37.Q0) GO TO 3112
         FACT=37.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(12) = 0.Q0
3112   CONTINUE
       IF (K(13) .NE. 41.Q0) GO TO 3113
         FACT=41.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(13) = 0.Q0
3113   CONTINUE
       IF (K(14) .NE. 43.Q0) GO TO 3114
         FACT=43.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(14) = 0.Q0
3114   CONTINUE
       IF (K(15) .NE. 47.Q0) GO TO 3115
         FACT=47.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(15) = 0.Q0
3115   CONTINUE
3000  CONTINUE
3999  CONTINUE
      C=PN/PD
      RETURN
      END
```

Program B was used to generate 25 random samples each of size 5 with one outlier, starting from a uniform (0,1) distribution. The samples of Weibulls are printed out as X's and transformed to W's, samples of EV_I, by a ℓn transformation. For each sample $\bar{w}$ and s_w^2 are computed. Also for each sample the smallest observation is replaced by the second smallest and by the largest and the new means $\bar{w}_{(2,1)}$, $\bar{w}_{(n,1)}$ and variances $s^2_{(2,1)}$, $s^2_{(n,1)}$ are computed. Also the largest observation is replaced by the second largest observation and by the smallest observation and again means $\bar{w}_{(n-1,n)}$, $\bar{w}_{(1,n)}$ and variances $s^2_{(n-1,n)}$, $s^2_{(1,n)}$ are computed for each sample.

Program B

```
C     This program computes Weibull and EV variables,
C     every fifth one spurious.
C     It prints out samples of size 5
C     along with mean and variance.
C     For each sample it also computes
C     the Winsorized mean and variance
C     by replacing the smallest
C     observation or the largest observation.
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION  X(125), W(125), W21(125), W51(125),
       W45(125), W15(125)
      DIMENSION WB(25), WV(25), W21B(25), W21V(25),
        W51B(25), W51V(25)
      DIMENSION W45B(25), W45V(25), W15B(25), W15V(25)
      REAL RAN(125)
      INTEGER ISEED(2)
      INTEGER*2 HSEED(3)
      REAL*8 DSEED
      EQUIVALENCE (ISEED(2),HSEED(1))
C
      WRITE(6,200)
 200  FORMAT(' Enter BETA and K')
      READ(5,100) BETA, C
 100  FORMAT(2F12.5)
      WRITE(7,201) BETA, C
 201  FORMAT('1       BETA = ',F12.5,'    K = ',F7.5)
      BETAC = BETA*C
      NSIZE = 5
      NSAMP = 25
      NTOTAL = NSIZE*NSAMP
C
C Get random number depending on time of day into DSEED
C
      CALL TIME(4,0,ISEED)
      HSEED(3) = HSEED(1)
      HSEED(1) = HSEED(2)
      HSEED(2) = HSEED(3)
      DSEED = DABS(DFLOAT(ISEED(2)))
C
C Call *IMSL routine GGUBS to get NTOTAL random
C   numbers into RAN
C
      NR = NTOTAL
      CALL GGUBS(DSEED, NR, RAN)
C
```

```
C Compute Weibull r.v.'s X and W
C
      DO 301 K=1,NTOTAL
        FACT = BETA
        IF (MOD(K,5) .EQ. 0)   FACT = BETAC
        X(K) = DBLE(-ALOG(RAN(K)))**(1.DO/FACT)
        W(K) = DLOG(X(K))
 301  CONTINUE
C
C Print out X and W
C
      WRITE(7,202)
 202  FORMAT('-           X'/)
      WRITE(7,203) X
 203  FORMAT(25(3X,5(1X,E13.6)/))
      WRITE(7,204)
 204  FORMAT('0           W')
      WRITE(7,203) W
C
C Compute Means and Variances for each sample of size
C NSIZE in X and W
C
      K = 1
      DO 310 KK=1,NSAMP
        CALL STAT(W, K, NSIZE, WB(KK), WV(KK))
        K = K+NSIZE
 310  CONTINUE
C
C Find the smallest, second smallest, largest, and second
C largest entry in each of the samples in W.
C
      K = 1
      DO 320 KK=1,NSAMP
        WMIN1 = 1.D75
        WMAX1 = -1.D75
        DO 321 K1=1,NSIZE
          IF (W(K) .GE. WMIN1)  GO TO 3211
            WMIN2 = WMIN1
            WMIN1 = W(K)
            KMIN = K
            GO TO 3212
 3211     CONTINUE
          IF (W(K) .GE. WMIN2)  GO TO 3212
            WMIN2 = W(K)
 3212     CONTINUE
```

```
C
          IF (W(K) .LE. WMAX1)  GO TO 3216
            WMAX2 = WMAX1
            WMAX1 = W(K)
            KMAX = K
            GO TO 3217
 3216     CONTINUE
          IF (W(K) .LE. WMAX2)  GO TO 3217
            WMAX2 = W(K)
 3217     CONTINUE
          K = K + 1
 321    CONTINUE
C
C Replace the smallest and largest entries with
C the second smallest
C and largest and with the second largest and
C smallest entries to
C form the altered samples W(21), W(51), W(45), and W(15).
C Recompute the mean and variance for each of
C the altered samples.
C
        K = K - NSIZE
        DO 322 K1=1,NSIZE
          W21(K) = W(K)
          IF (K .EQ. KMIN)  W21(K) = WMIN2
          W51(K) = W(K)
          IF (K .EQ. KMIN)  W51(K) = WMAX1
          W45(K) = W(K)
          IF (K .EQ. KMAX)  W45(K) = WMAX2
          W15(K) = W(K)
          IF (K .EQ. KMAX)  W15(K) = WMIN1
          K = K + 1
 322    CONTINUE
 320  CONTINUE
C
      K = 1
      DO 330 KK=1,NSAMP
        CALL STAT(W21, K, NSIZE, W21B(KK), W21V(KK))
        CALL STAT(W51, K, NSIZE, W51B(KK), W51V(KK))
        CALL STAT(W45, K, NSIZE, W45B(KK), W45V(KK))
        CALL STAT(W15, K, NSIZE, W15B(KK), W15V(KK))
        K = K + NSIZE
 330  CONTINUE
      WRITE(7,225)
```

```
 225  FORMAT('$**$FORMAT=FMTL1 FONTNEXTIMAGE=
        1200.MEDIUM.9.FIXED.LANDSCAPE.1')
      WRITE(7,250)
 250  FORMAT('1 Sample     WBAR
               WVAR      W(21)BAR W(21)VAR   W(51)
     BAR W(51)VAR   W(45)BAR W(45)VAR
      W(15)BAR W(15)VAR
      DO 350 J=1,NSAMP
C
C Reassign these so the WRITE statement will
C fit on one line.
C
        W1 = WB(J)
        W2 = WV(J)
        W3 = W21B(J)
        W4 = W21V(J)
        W5 = W51B(J)
        W6 = W51V(J)
        W7 = W45B(J)
        W8 = W45V(J)
        W9 = W15B(J)
        W10 = W15V(J)
        WRITE(7,255) J, W1, W2, W3, W4, W5, W6,
            W7, W8, W9, W10
 255    FORMAT(6X,I2,2X,5('  ',2(F8.5,1X)),'')
 350  CONTINUE
      STOP
      END
      SUBROUTINE STAT(X, KK, NSIZE, XBAR, XVAR)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(125)
      KSTOP = KK + NSIZE - 1
      SUM = 0.0D0
      SSQ = 0.0D0
      DO 300 K=KK,KSTOP
        SUM = SUM + X(K)
        SSQ = SSQ + (X(K)**2)
 300  CONTINUE
      XBAR = SUM/DFLOAT(NSIZE)
      XVAR = (SSQ - ((SUM**2)/DFLOAT(NSIZE)))/
             (DFLOAT(NSIZE) - 1
      RETURN
      END
```